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Operator Methods
For Piecewise-Linear Network Analysis

by
C. H. Roth

Technical Report No. 2056-1
31 October 1961

STANFORD ELECTRONICS LABORATORIES

STANFORD UNIVERSITY · STANFORD, CALIFORNIA

OPERATOR METHODS FOR PIECEWISE-LINEAR NETWORK ANALYSIS

by

C. H. Roth
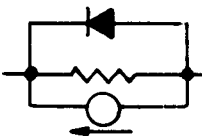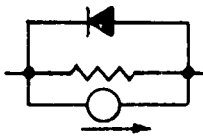
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Stanford Electronics Laboratories

Stanford University   Stanford, California

Page		Is	Should Be
3	Fig. 1	SLOPE = g_2	SLOPE = $-g_2$
7	Fig. 5(a)		
15	Fig. 11(b)	$(i_1 - 2v_1) > 0$	$(i_1 - 2v_1) > 7/2$
29	Last Eq. first ( )	$\begin{pmatrix} 1, & -1/2 \\ -1, & -1 \\ -2, & -1/3 \end{pmatrix}^{-1}$	$\begin{pmatrix} 1, & -1/2 \\ -1, & -1 \\ -1/2, & -1/3 \end{pmatrix}^{-1}$
36	Fig 19	v_a	v_2
40	2nd Eq. first ( )	$\begin{pmatrix} 0,2 \\ 0,-5 \end{pmatrix}$	$\begin{pmatrix} 0,2 \\ 0,.5 \end{pmatrix}$
49	Fig. 21	$y_1\ y_2$	$y_1,\ y_2$
54	Eq.	6^*	4^*
74	Line 2	$\begin{pmatrix} \dfrac{-B_1}{b_2}, & \dfrac{1}{b_2} \end{pmatrix}$	$\begin{pmatrix} \dfrac{-b_1}{b_2}, & \dfrac{1}{b_2} \end{pmatrix}$
	Eq. (82)	$(X_{k+1} - X_k)$	$(X_{k+1} - X)$
78	Eq. (90)	$X_2 \begin{pmatrix} 0,1 \\ 0,0 \end{pmatrix} (i_1)$	$X_2 \begin{pmatrix} 0,0 \\ 0,1 \end{pmatrix} (i_1)$
82	Eq. (98)	$-\overline{B}*\overline{A}*-C$	$-(\overline{B}*\overline{A}*-C)$
89	Last Eq	$\begin{pmatrix} -1,3/2 \\ -1,\ 2 \\ 1,\ 1 \end{pmatrix} (R_1 R_2^{-1}+I)^{-1}$	$\begin{pmatrix} -1,3/2 \\ -1,\ 2 \\ -1,\ 1 \end{pmatrix}, (R_1 R_2^{-1} + I)^{-1}$
97	Eq. (130)	$-z_{32}(R_2 + z_{22})^{-1} z_{31}$	$-z_{32}(R_2 + z_{22})^{-1} z_{21}$
118	Fig. 49(b)	$i_2 < 2 \quad i_2 > 2$	$i_1 < 2 \qquad i_1 > 2$

Page		Is	Should Be
121	Line 2	$(\frac{2}{3}v_1 - \frac{1}{3}v_2)$	$(\frac{2}{3}v_1 - \frac{1}{3}v_2)$,
	Line 3	$+\begin{pmatrix} 0,0 \\ 0,1 \end{pmatrix}$	$\begin{pmatrix} 0,0 \\ 0,1 \end{pmatrix}$
131	Fig. 54(b)	$(i_e + ai_e)$	$(i_c + ai_e)$
163	2nd line from bottom	$|y_1| < \epsilon,$	$|y_1| < \epsilon$
171	Line after Eq.(a-6)	F_i's	f_i's

ABSTRACT

The object of this research is the development of systematic meth-
ods for the analysis of networks which contain piecewise-linear (PWL)
elements. Many physical devices have nonlinear characteristics that can
be closely approximated by PWL characteristics. In order to facilitate
the systematic analysis of networks that contain such devices, "PWL
operators" have been introduced to represent the characteristic curves
of PWL elements.

Rules for addition, subtraction, multiplication, inversion, and
other operations with PWL operators have been formulated, and the alge-
braic properties of these operations have been studied. Addition of PWL
operators is associative and commutative, and multiplication is associa-
tive but not commutative. Methods for solving PWL-operator equations
have also been investigated.

The input and transfer characteristics of resistive PWL ladder net-
works can easily be calculated in terms of PWL operators. A general
procedure has been formulated for analysis of any network that contains
an arbitrary configuration of linear resistors and two PWL resistors.
Analysis of resistive PWL networks with three PWL resistors leads to an
equation of the form

$$A(X + I) = BX + C$$

where A, B, and C are known PWL operators, I is the identity operator,
and X is an unknown PWL operator. Because the distributive law for PWL
operators does not hold from the left, this equation cannot be solved
in terms of the basic algebraic operations; therefore, a new operation
called "trivolution" has been defined to solve this equation.

PWL operators are useful in the analysis of electronic circuits.
For large-signal operation, the characteristics of diodes, vacuum tubes,
transistors, and other electronic devices can be approximated by PWL
characteristics and described in terms of PWL operators. PWL input and
transfer characteristics of electronic circuits can be determined by
this method.

To describe the behavior of PWL networks which contain energy storage elements, PWL differential equations can be formulated in terms of PWL operators. For PWL R-C, R-L, and L-C network problems, solutions to these differential equations can be expressed in terms of PWL operators. For example, the voltage across a parallel combination of a PWL resistor and a PWL capacitor can be expressed as a PWL operator operating on an exponential with a PWL exponent. Further development of methods for solving PWL differential equations is needed.

The use of PWL operators provides a convenient way of solving PWL network problems on a digital computer. Programs have been written for the Burroughs 220 Computer for carrying out algebraic operations with PWL operators and for solving PWL-operator equations. Iterative methods have been developed for computer solution of higher-order PWL equations and sets of simultaneous PWL equations that cannot be solved directly in terms of the basic algebraic operations.

Since any nonlinear element can generally be approximated by a PWL characteristic within any desired degree of accuracy, it is hoped that the PWL operator method which has been developed will have fairly wide application to the approximate analysis of nonlinear networks.

CONTENTS

TABLES

ILLUSTRATIONS

I. INTRODUCTION

The object of this research has been to develop systematic methods for the analysis of networks that contain piecewise-linear elements. A general method has been developed for analysis of resistive PWL[*] networks, and special cases of PWL networks containing reactive elements have been solved. One reason for studying PWL-network theory is that many electronic devices such as diodes, transistors, and vacuum tubes have characteristics which can be closely approximated by PWL curves. Furthermore, almost any nonlinear characteristic can be approximated to any desired degree of accuracy by a suitably chosen PWL characteristic. Although PWL networks are more difficult to analyze than linear networks, they are easier to analyze than more general nonlinear networks. For this reason, PWL-network theory is useful for the approximate analysis of nonlinear networks. The study of PWL networks is a compromise between studying linear networks, which have already been extensively studied, and nonlinear networks, which are very difficult to analyze.

In general, it is possible to analyze a PWL network on a section-by-section basis. This method reduces the solution to a series of linear problems, each of which can be solved by standard linear-circuit techniques. After solution of each linear problem, one must determine which PWL element will next change to a new section of its PWL characteristic and then formulate the appropriate boundary conditions for the next linear problem. When the number of PWL elements is large, or when each PWL characteristic curve has a large number of sections, solution on the section-by-section basis becomes very tedious. The need for a more systematic method for analysis of PWL networks is thus apparent. A systematic method of analysis and a concise notation are also needed in order to utilize high-speed digital computers efficiently for the solution of PWL networks.

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[*] PWL will be used as an abbreviation for piecewise-linear.

A. PWL-NETWORK MODELS

The use of idealized models is necessary in the analysis of physical systems. Such models abstract the essential properties of the physical system, but at the same time they are simple enough to be useful for purposes of analysis. For a given physical device, different models can be constructed to serve different purposes. The type of model that is used often depends on the amplitude and frequency of the input to the system. For small-signal analysis, a linear model may be used; but for different regions of operation, different linear models may be required. For operation over a large range of inputs, a linear model is often inadequate and a nonlinear or a PWL model is needed.

The nonlinear i-v characteristic of a typical tunnel diode [Ref. 1] is shown in Fig. 1. For small-signal operation about the point v_o, the linear model of Fig. 2 may be used, with $-g_o$ equal to the slope of the i-v curve at the operating point. For large-signal operation, the i-v characteristic may be approximated by a five-section PWL curve as shown. Either fewer or more sections may be used in the PWL approximation, depending on the accuracy which is needed.

Circuit models for PWL devices can be constructed using ideal diodes, linear resistors, and sources. The ideal diode, whose v-i characteristic is shown in Fig. 3, has two states. When $i > 0$, $v = 0$ and the diode is "on"; when $v \leq 0$, $i = 0$ and the diode is "off". Fig. 4 shows one possible circuit model for the PWL tunnel diode characteristic of Fig. 1. In this model, diode D_1 conducts for $v < v_1$, D_2 for $v < v_2$, D_3 for $v > v_3$, and D_4 for $v > v_4$.

B. METHODS FOR ANALYSIS OF PWL NETWORKS

There are two basic approaches to the analysis of networks that contain resistive PWL elements. In the first approach, a circuit model composed of ideal diodes, resistors, and sources is constructed for each PWL element, and the resulting resistive-diode network is analyzed by considering the states of the individual diodes. In the second approach, the characteristic curves of the PWL elements are represented symbolically, and the analysis is carried out directly in terms of this symbolism.
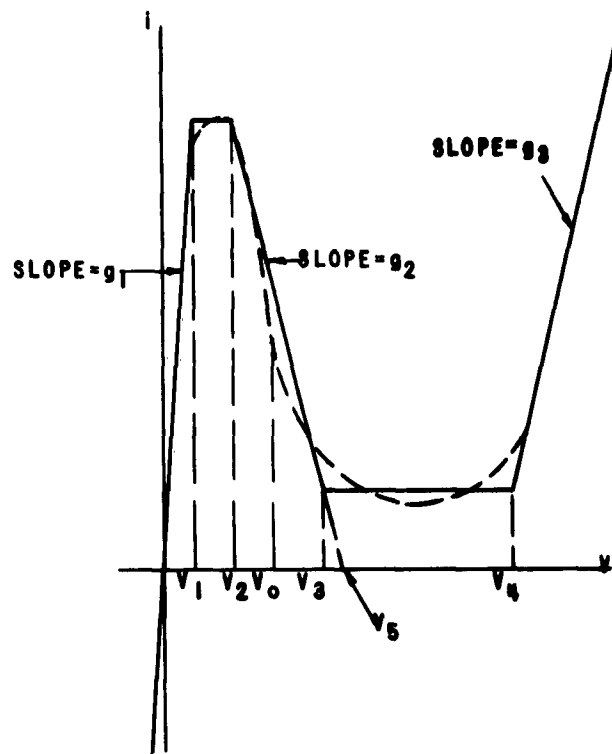
FIG. 1. PWL APPROXIMATION TO TUNNEL
DIODE CHARACTERISTIC.
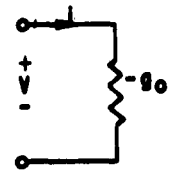
FIG. 2. LINEAR MODEL
FOR TUNNEL DIODE.
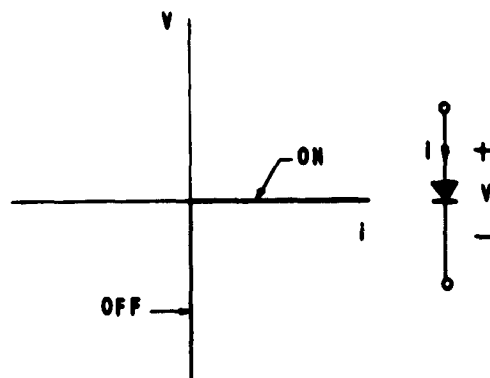
FIG. 3. IDEAL DIODE CHARACTERISTIC.
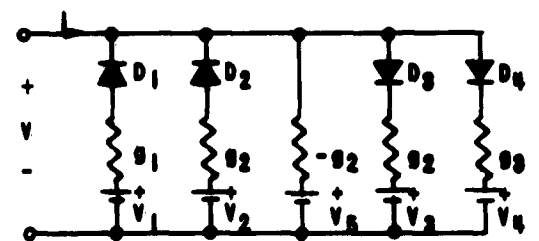
FIG. 4. PWL MODEL FOR TUNNEL DIODE

- 3 -

When the latter approach is used, the necessity for drawing diode models is eliminated; therefore, it is unnecessary to consider the individual diode states during the process of analysis.

There are two conventional methods for analyzing resistive diode networks--the method of assumed states and the breakpoint method. These methods are discussed in detail with many examples in Electronic Circuit Theory [Ref. 2]. In the method of assumed states, the circuit is analyzed using all possible combinations of diode states. For each assumed set of diode states, each conducting diode is replaced with a short circuit, each nonconducting diode is replaced with an open circuit, and the circuit reduces to a network of linear resistors and sources. The v-i characteristic of each reduced network is a straight line, and the PWL characteristic of the original network must consist of portions of these lines. The appropriate portions of the lines can usually be determined by considering what happens in the network as the terminal voltage or current is varied. If the network contains n diodes, there are 2^n possible combinations of diode states; consequently, the amount of work required for this method increases rapidly with the size of the network. Furthermore, much of this work may be wasted because many of the possible combinations of diode states may never actually occur for any input voltage.

The breakpoint method is usually more efficient than the method of assumed states. The points at which successive line segments of a PWL curve meet are called breakpoints. A PWL curve is completely determined by specifying the coordinates of its breakpoints and the slope of both end segments. Each breakpoint on the characteristic curve of a diode network corresponds to a change of state of one of the diodes. At the breakpoint of a diode, the current and voltage are both zero for that diode. This constraint determines the input voltage and current to the network at the breakpoint if the states of the other diodes are known. The breakpoints can be determined successively by considering what happens in the network as the terminal voltage or current is increased.

An algebraic method for analysis of simple diode networks was presented by Schaefer [Ref. 3] in 1954. Stern [Refs. 4,5] developed an improved and more-general version of this algebraic method in 1956. In Stern's method, the characteristics of PWL elements are expressed

symbolically in terms of ϕ^+ and ϕ^- transformations (see Section II. E). To analyze a network, loop or node equations can be written in terms of these transformations. By applying the rules of Stern's algebra, these equations can be solved for the desired PWL characteristic. Stern's method is discussed further in Appendix A and in Section X. B. Dennis [Ref. 6] gives a procedure for tracing the PWL curve of a resistive diode network. This procedure is related to the methods of quadratic programming used in operations research.

C. THE PWL-OPERATOR METHOD

The existing methods for analysis of PWL networks have been briefly discussed. As an attempt to find a more systematic method that is suitable for use with a digital computer, the PWL-operator method has been developed. In Chapter II, PWL operators are defined to represent the characteristic curves of PWL elements. The basic algebraic operations are defined for PWL operators and the algebraic properties of these operations are studied in Chapter III. Chapter IV introduces a new operation, which solves a class of PWL-operator equations that cannot be solved in terms of the basic algebraic operations. In Chapter V, PWL-operator methods are applied to determine input and transfer characteristics of resistive PWL networks, and in Chapter VI, the analysis of PWL two-ports is considered. PWL-operator methods are used to analyze vacuum-tube and transistor circuits in Chapter VII. Extension of PWL-operator methods to PWL networks that contain reactive elements is considered in Chapter VIII. Chapter IX discusses computer programs for the analysis of PWL networks.

The PWL-operator methods are not intended to be mathematically rigorous in all cases, but rather are intended to be practical methods of solving problems. Instead of being concerned with rigorous definitions and proofs of theorems, for the most part, we will deal with the development of the theory and its practical application. Although an attempt has been made to make the theory as generally applicable as possible, exceptional "pathological" examples occur where an invalid solution is occasionally obtained. Since these pathological cases generally correspond to non-physically-realizable situations, we are not concerned with them as engineers, and we will let the mathematicians worry about them.

II. MATHEMATICAL REPRESENTATION OF PWL CURVES

The first step in developing a systematic method for analysis of PWL networks is to determine a concise mathematical representation for PWL curves. After examination of some of the types of PWL curves that can be encountered in the analysis of resistive PWL networks, PWL operators are defined to represent such curves. Stern's notation for representing PWL functions is also discussed.

A. CHARACTERISTIC CURVES OF RESISTIVE PWL NETWORKS

A PWL curve consists of a series of adjoining line segments. PWL curves can be classified according to their range of definition. They can be defined for (1) an infinite, (2) a semi-infinite, or (3) a finite range of values of the independent variable. Furthermore, two or more separate curves of these types may be combined to form a composite characteristic curve. Network models composed of ideal diodes, positive and negative[*] resistances, and independent and dependent sources will be used to illustrate some of the types of PWL characteristic curves that can occur.

The first type of PWL curves, which are defined for all values of the independent variable, occur most frequently. Examples of networks having this type of v-i characteristic are given in Figs. 5 and 6.[**] The ladder network of Fig. 5a is easily analyzed by the breakpoint method. The results of this analysis are given in Figs. 5b and 5c. The states of the ideal diodes are indicated by 1's and 0's, where "1" indicates that the diode is on, and "0" indicates that the diode is off.

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[*] A negative resistance, -R, is equivalent to a positive resistance in parallel with a dependent current source as shown:
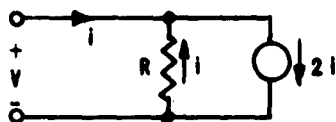
[**] For further discussion of negative resistance, see [Ref. 2, pp. 437-438]. All numerical values given on illustrations will be in ohms, volts, and amperes unless otherwise specified.
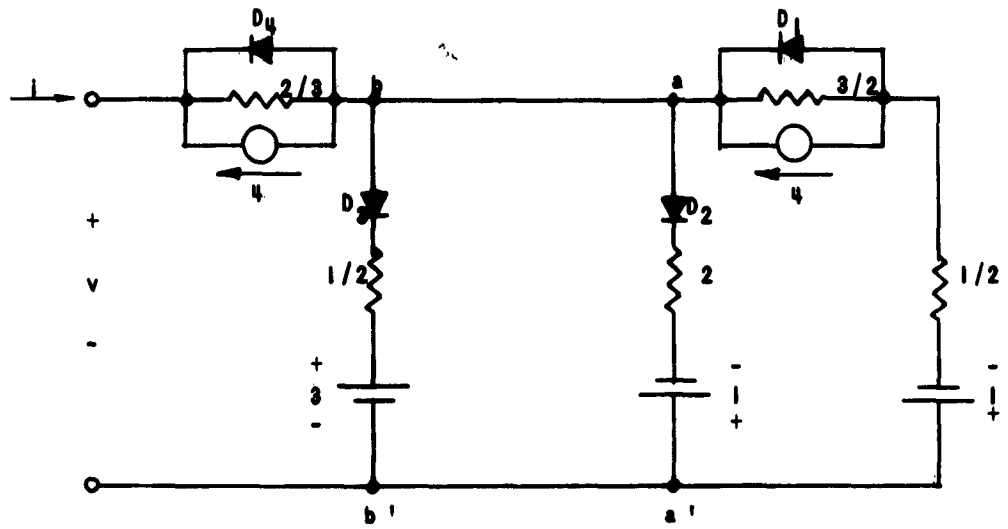
(a)
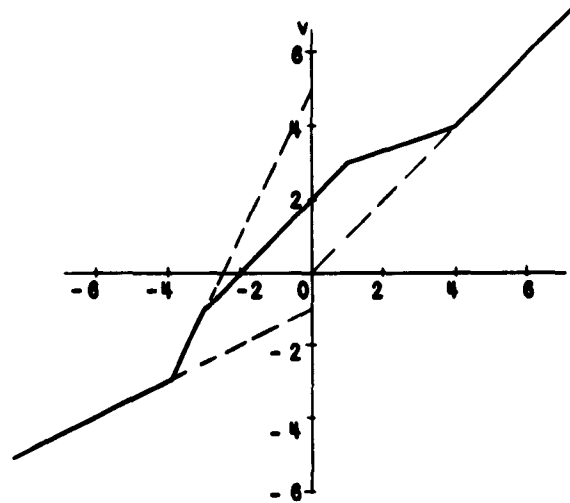
(b)

DIODE STATES				RANGE OF CURRENT	VOLTAGE
D_4	D_3	D_2	D_1		
1	0	0	1	$i < -4$	$v = -i + 1/2$
1	0	0	0	$-4 < i < -3$	$v = 5 + 2i$
1	0	1	0	$-3 < i < i$	$v = 2 + i$
1	1	1	0	$i < i < 4$	$v = 8/3 + i/3$
0	1	1	0	$4 < i$	$v = 0 + i$

(c)

FIG. 5. LADDER NETWORK AND ITS V-I CHARACTERISTIC.
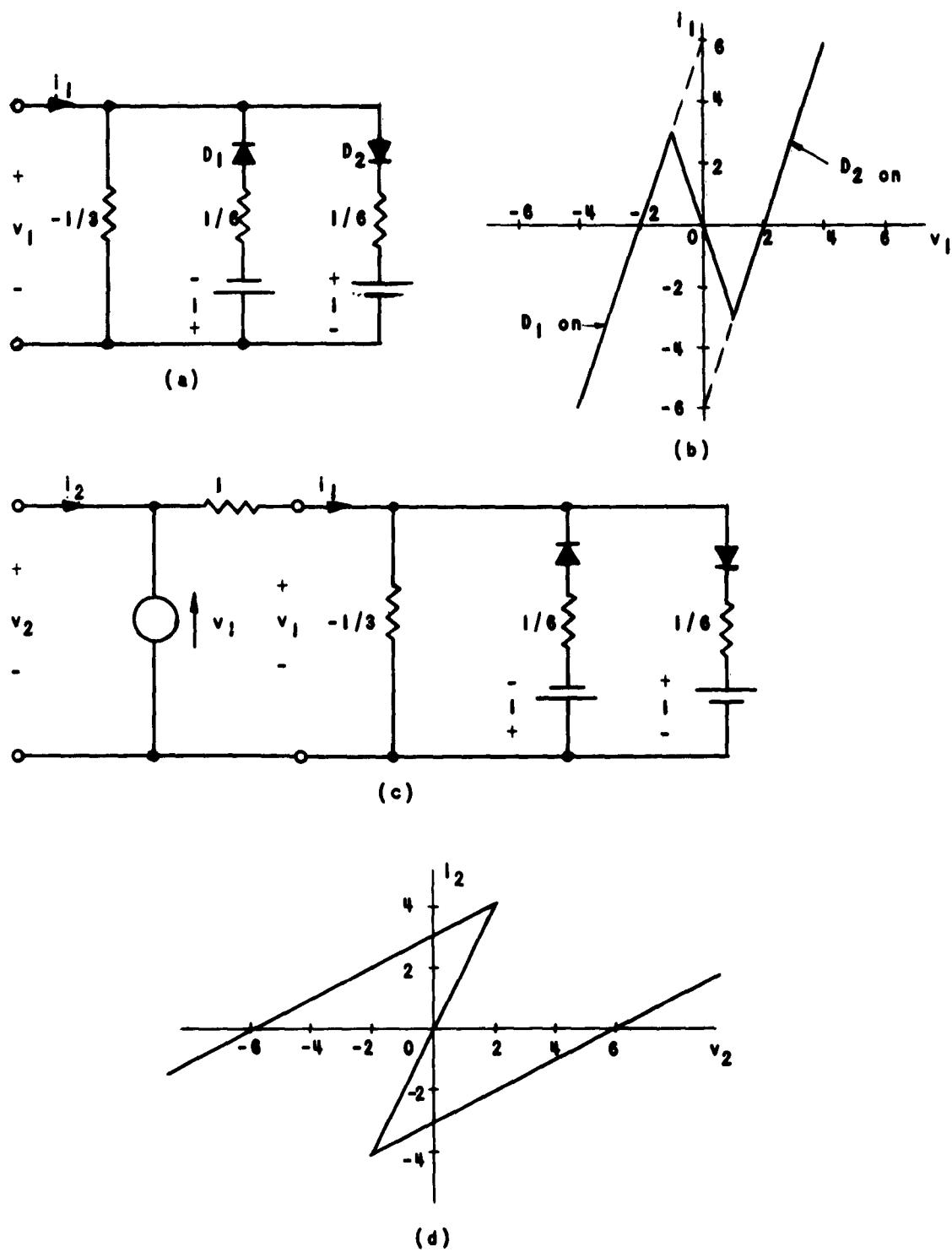
FIG. 6. NETWORKS WITH MULTI-VALUED PWL CHARACTERISTICS.

The network of Fig. 6a has an i-v characteristic that is single-valued, but the corresponding v-i characteristic is multi-valued. Adding a dependent current source of value v_1 and a 1-ohm resistor to this network yields the network of Fig. 6c. The added network brings about the following transformation:

$$i_2 = i_1 - v_1 = \sqrt{2}\,(i_1\cos 45^\circ - v_1\sin 45^\circ)$$

$$v_2 = i_1 + v_1 = \sqrt{2}\,(i_1\cos 45^\circ + v_1\sin 45^\circ)$$

This transformation rotates the i-v characteristic (Fig. 6b) by 45° and changes the scale by a factor of $\sqrt{2}$, yielding the new i-v characteristic (Fig. 6d). This new curve is multi-valued in both variables.

Examples of the second type of PWL characteristic, which is defined for a semi-infinite range of values of the independent variable, are given in Fig. 7. The network of Fig. 7a has a v-i characteristic (Fig. 7b) that is defined only for $i \geq 0$. When the diode is off, $v = -i$, while when the diode is on, $v = +i$. To verify that the characteristic is undefined for $i < 0$, let $i = -i_o$. If we assume that the diode is on, $v = -i_o$, which implies that the voltage across the diode is negative, and the diode is off. On the other hand, if we assume that the diode is off, $v = +i_o$, which implies that the voltage across the diode is positive and the diode is on. Since either assumption leads to a contradiction, we can only conclude that the characteristic curve is undefined for $i < 0$. At this point, one may be tempted to inquire what would happen if we actually built the network and placed a negative current source across its terminals. The difficulty here results from over-idealization of the network model. In practice, it is impossible to build a device which has a negative-resistance characteristic over an infinite range of voltage since such a device would have to be capable of supplying infinite power. Eventually, for sufficiently large values of voltage, the negative-resistance device must have positive resistance, which implies that the characteristic of Fig. 7b must eventually double back if it is to represent a physically realizable network.
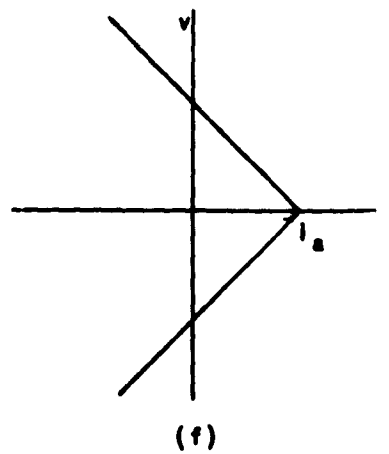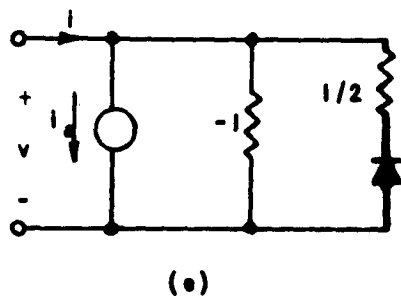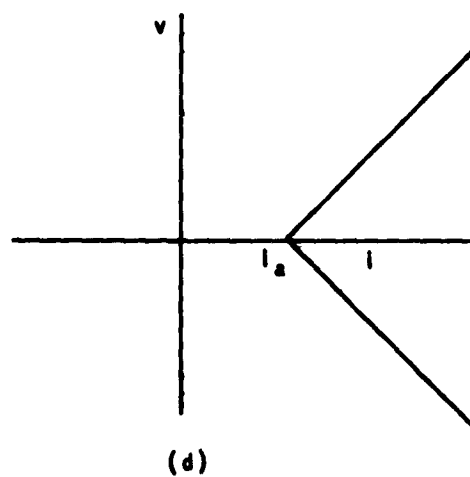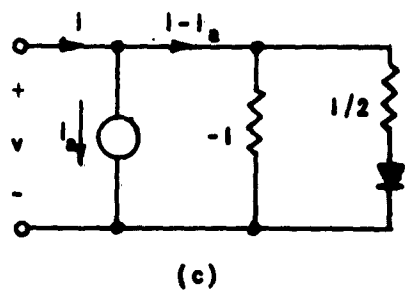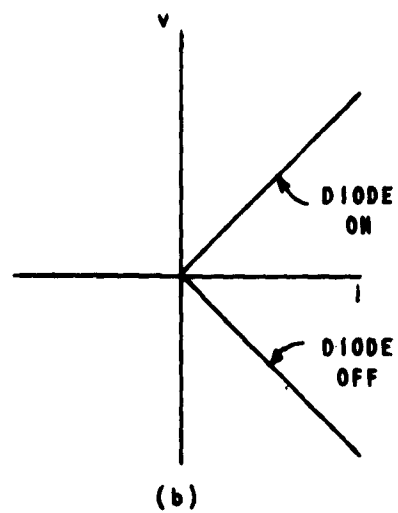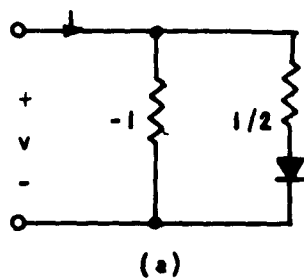
FIG. 7. NETWORKS WITH V-I CHARACTERISTICS DEFINED FOR A SEMI
INFINITE RANGE OF CURRENT.

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Another embarrassing question one might ask about the characteristic of Fig. 7b is which branch of the curve would be followed if the input current were increased starting with $i = 0$. In order to answer this question, we must again modify our ideal model and consider the presence of parasitic elements. The difficulty can be resolved if we place a stray capacitance across the input terminals of the network. When the current is increased, it will initially flow into the capacitor, starting to charge it to a positive voltage. The diode will then start conducting, which makes the circuit look like a 1-ohm resistor in parallel with the capacitor, and the upper part of the curve will be followed as the current is increased.

Placing a current source of value i_a in parallel with the network of Fig. 7a, as shown in Fig. 7c, gives a characteristic curve (Fig. 7d) that is defined only for $i > i_a$. Reversing the diode (Fig. 7e) gives a characteristic (Fig. 7f) defined only for $i < i_a$.

Examples of the third type of PWL characteristic, which is defined for only a finite range of values of the independent variable, are given in Figs. 8 and 9. The network of Fig. 8a consists of the network of Fig. 7a in series with the network of Fig. 7e with $i_a = 1$. Since the current is the same in both series networks, the composite characteristic is obtained by adding voltages that correspond to the same value of current on the characteristics of Figs. 7b and 7f. Since adding something undefined to something defined yields something undefined, the resulting characteristic curve, Fig. 8b, is undefined for $i < 0$ or $i > 1$. In addition to being a closed loop, this characteristic has the interesting property that it crosses itself without actually intersecting. It is impossible to go directly from segment AC to segment BD at the crossover point ($v = 0$ and $i = \frac{1}{2}$) because each segment corresponds to a different state of the diodes (Fig. 8c) and there is no external means of causing the diodes to change state while keeping both v and i constant. Furthermore, when we are operating at the crossover point, just specifying v and i is insufficient to tell us what the internal state of the network is.

The network of Fig. 9a, which contains a dependent current source and a negative resistance, has a v-i characteristic which is a closed parallelogram (Fig. 9d). When $i_1 > 0$, diode D_2 is on, and the network reduces

(a)

(b)
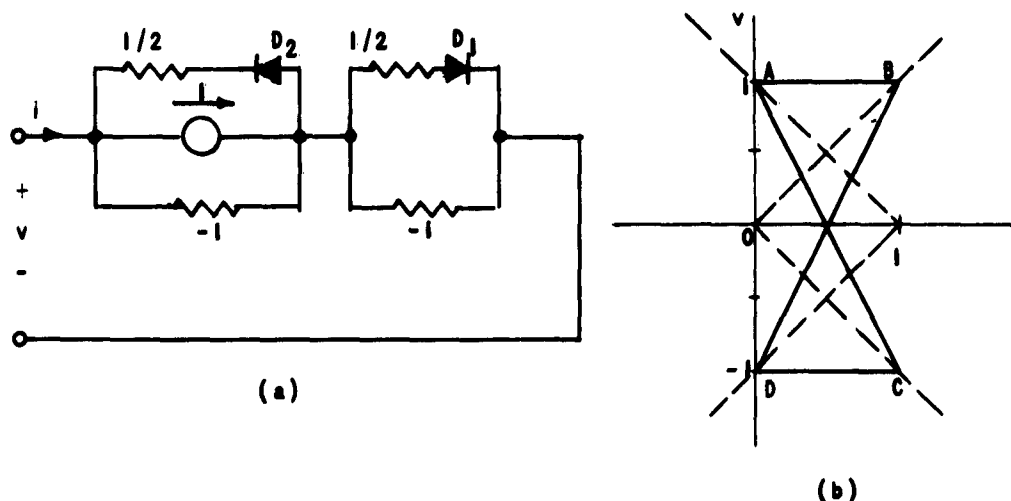
Segment	Diode States D_2	Diode States D_1	Range of current	Voltage		
AC	0	0	$0 \leq i \leq 1$	$v = (1-i)$	$+ (-i) =$	$1-2i$
CD	1	0	$0 \leq i \leq 1$	$v = (-1+i)$	$+ (-i) =$	-1
DB	1	1	$0 \leq i \leq 1$	$v = (-1+i)$	$+ \quad i =$	$-1+2i$
BA	0	1	$0 \leq i \leq 1$	$v = (1-i)$	$+ \quad i =$	1

(c)

FIG. 8. NETWORK WITH ITS V-I CHARACTERISTIC DEFINED FOR A FINITE RANGE OF CURRENT.

to Fig. 9b. When $i_1 < 0$, D_2 is off, and the network reduces to Fig. 9c after transforming the dependent source and canceling the negative resistance. The characteristic of Fig. 9d can then be derived from Fig. 9b and 9c, noting that D_1 is off when $(i_1 - 2v_1) < 0$ and D_1 is on when $(i_1 - 2v_1) > 0$.

Examples of networks whose characteristic curves have two distinct branches are given in Figs. 10 and 11. The characteristics of the two subnetworks which compose the network of Fig. 10a are shown as dashed lines in Fig. 10b. Since the subnetworks are in series, the overall v-i characteristic is obtained by adding voltages which correspond to the same value of current on the dashed curves. The resulting curve has two distinct branches, and the v-i characteristic is undefined for i > 0 or -1 < v < +1.

Addition of a diode, resistor, and current source to the network of Fig. 9a yields the network of Fig. 11a. The v-i characteristic of this
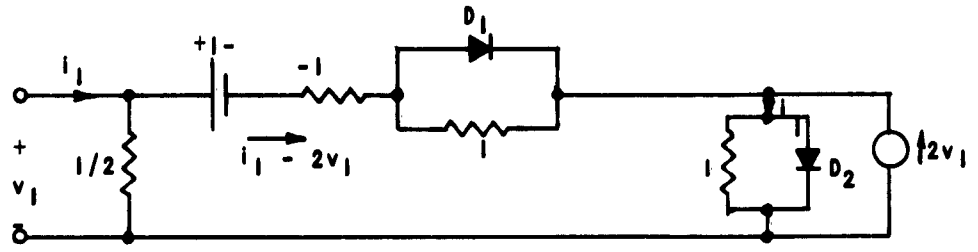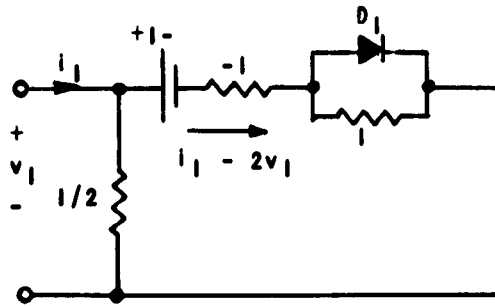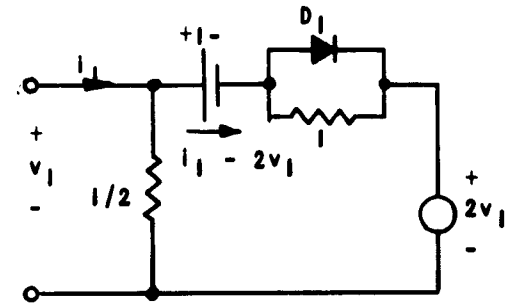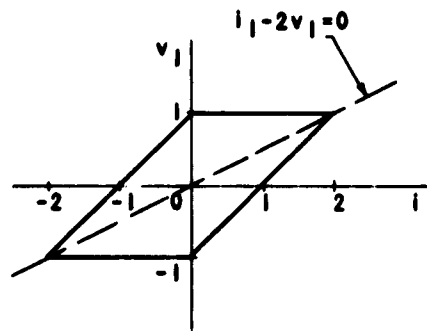
| DIODE STATES | | RANGE OF | VOLTAGE (v) |
D_2	D_1	CURRENT (AMP)	
0	0	$-2 \leq i_1 \leq 0$	$v_1 = 1 + i_1$
0	1	$-2 \leq i_1 \leq 0$	$v_1 = -1$
1	1	$0 \leq i_1 \leq 2$	$v_1 = i_1 - 1$
1	0	$0 \leq i_1 \leq 2$	$v_1 = 1$

(e)

FIG. 9. NETWORK WITH A CLOSED-LOOP V-I CHARACTERISTIC.

network has two distinct branches (Fig. 11c). When $(i_1 - 2v_1) < 7/2$, D_3 is on, and the network reduces to Fig. 9a, which has the closed-loop v-i characteristic shown above the dashed line in Fig. 11c. When $(i_1 - 2v_1) > 7/2$, D_3 is off, and the network reduces to Fig. 11b, which has the monotonic v-i characteristic shown below the dashed line.

The above examples illustrate some of the many types of v-i characteristics that resistive PWL networks can have. Any method of analysis that is to be generally applicable to PWL networks containing controlled sources or negative resistances must be capable of handling characteristic curves that are multi-valued and that may be undefined for some regions of the variables. Before developing methods of analysis, mathematical representations of PWL curves will be formulated.

B. REPRESENTATION OF PWL CURVES BY PWL OPERATORS

A straightforward way to describe a PWL curve is to list the linear equation for each section of the curve, together with the range over which this equation is valid. This method of description is adequate for all types of PWL curves, including multi-valued curves. Examples of this type of representation are given in Figs. 5c and 9e.

A PWL curve with n linear sections, which is defined for all values of x, can be specified by the following equations:

$$
\begin{aligned}
y &= q_1 + r_1 x & & (x \leq b_1) \\
y &= q_2 + r_2 x & & (x \text{ between } b_1 \text{ and } b_2) \\
&\ \ \vdots \\
y &= q_k + r_k x & & (x \text{ between } b_{k-1} \text{ and } b_k) \\
&\ \ \vdots \\
y &= q_n + r_n x & & (x \geq b_{n-1})
\end{aligned} \tag{1}
$$

where q_k is the y-intercept of the kth section,

$\quad$ r_k is the slope of the kth section, and

$\quad$ b_k is the abcissa of the intersection of the kth section and the $(k+1)^{th}$ section.

The equations of the sections are listed in the order in which they occur as the PWL curve is traced out, starting at the left. When $x = b_k$,
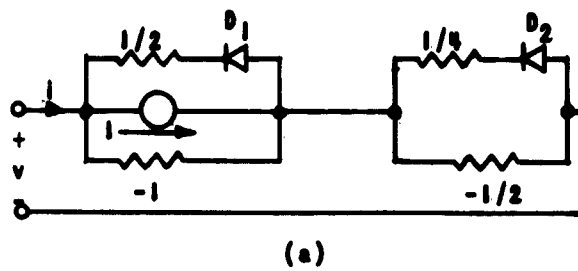
FIG. 10. NETWORK WITH A V-I CHARACTERISTIC
HAVING TWO DISTINCT BRANCHES.
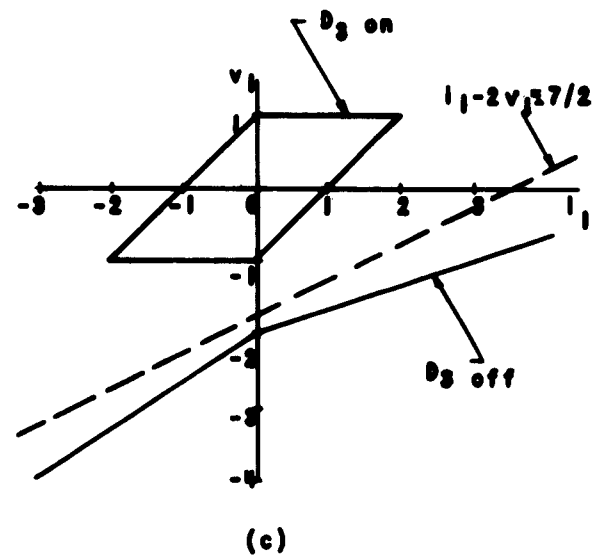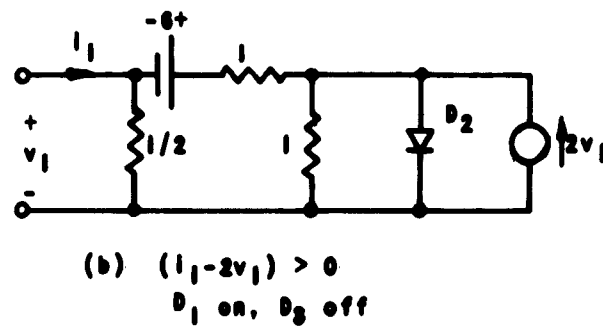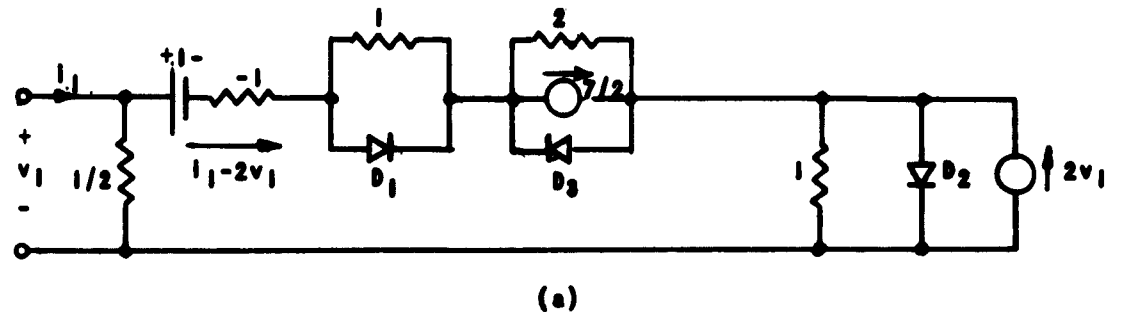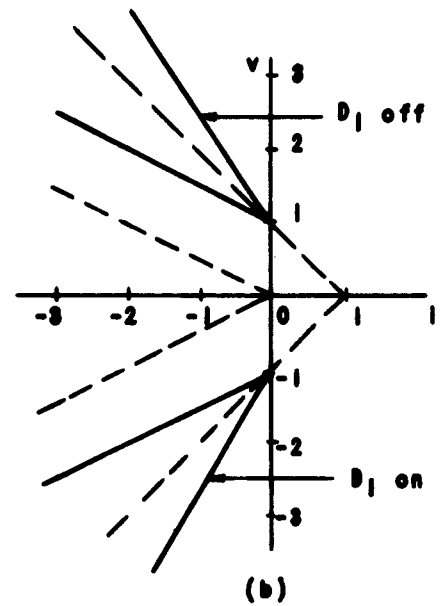
FIG. 11. NETWORK WITH A V-I CHARACTERISTIC HAVING TWO DISTINCT BRANCHES.

$$y = q_k + r_k b_k = q_{k+1} + r_{k+1} b_k \qquad (2)$$

Solving for the kth breakpoint yields

$$b_k = \frac{q_{k+1} - q_k}{r_k - r_{k+1}} \qquad (3)$$

Since the breakpoints can be calculated in terms of the slopes and intercepts, it is unnecessary to specify the breakpoints separately, and the above set of equations can be written in the following abbreviated form

$$y = \begin{pmatrix} q_1, & r_1 \\ q_2, & r_2 \\ \vdots & \\ q_k, & r_k \\ \vdots & \\ q_n, & r_n \end{pmatrix} (x) = A(x) \qquad (4)$$

The array of n intercepts and slopes, A, will be called an nth order PWL operator*. A PWL operator can be thought of as a concise mathematical representation of a PWL curve. A PWL operator contains the minimum amount of information necessary to describe a PWL curve since two pieces of information are necessary to determine each line segment. The only restrictions on the PWL curve are that it be defined for all values of the independent variable and that all of the slopes be finite.** The curve may have negative slopes, may be multi-valued, or even may intersect itself.

Given a PWL curve or the equations which describe it, one can find the corresponding PWL operator; or given a PWL operator, one can find the corresponding PWL curve and the equations which describe it. To determine the PWL operator which represents a given PWL curve, start with the leftmost linear section, follow the curve, and write down the intercept and slope of each section in the order in which the sections are encountered. For example, the curve of Fig. 5b is represented by the PWL operator

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$$A = \begin{pmatrix} -1, & \frac{1}{2} \\ 5, & 2 \\ 2, & 1 \\ 8/3, & 1/3 \\ 0, & 1 \end{pmatrix} \tag{5}$$

To reconstruct a PWL curve from a PWL operator, one possible procedure is:

1. Calculate the breakpoints by Eq. (3).
2. Calculate the value of y at each breakpoint by Eq. (2).
3. Plot these points and join each <u>successive</u> pair of points by a line segment.
4. Draw a line with slope r_1 starting at the left of b_1 and terminating at b_1.
5. Draw a line with slope r_n starting at the right of b_{n-1} and terminating at b_{n-1}.

The above procedure determines a unique PWL curve from a given PWL operator. For example, if

$$B = \begin{pmatrix} 3, & \frac{1}{2} \\ 3, & -\frac{1}{4} \\ 0, & -1 \\ -6, & 2 \\ -2, & 1 \end{pmatrix} \tag{6}$$

the breakpoints are calculated by Eq. (3) as

$$b_1 = 0, \quad b_2 = -4, \quad b_3 = 2, \quad b_4 = 4$$

The values of y at the breakpoints are calculated by Eq. (2) as

$$y_1 = 3, \quad y_2 = 4, \quad y_3 = -2, \quad y_4 = 2$$

---

[*] Capital letters will be used to designate PWL operators and small letters will be used for variables and constants throughout this report.

[**] Modified forms of PWL operators will be introduced later in order to eliminate these restrictions.

The resulting PWL curve (Fig. 12) can be described by the equations

$$y = 3 + x/2 \qquad x \leq 0$$
$$y = 3 - x/4 \qquad -4 \leq x \leq 0$$
$$y = \quad - x \qquad -4 \leq x \leq 2$$
$$y = -6 + 2x \qquad 2 \leq x \leq 4$$
$$y = -2 + x \qquad 4 \leq x$$
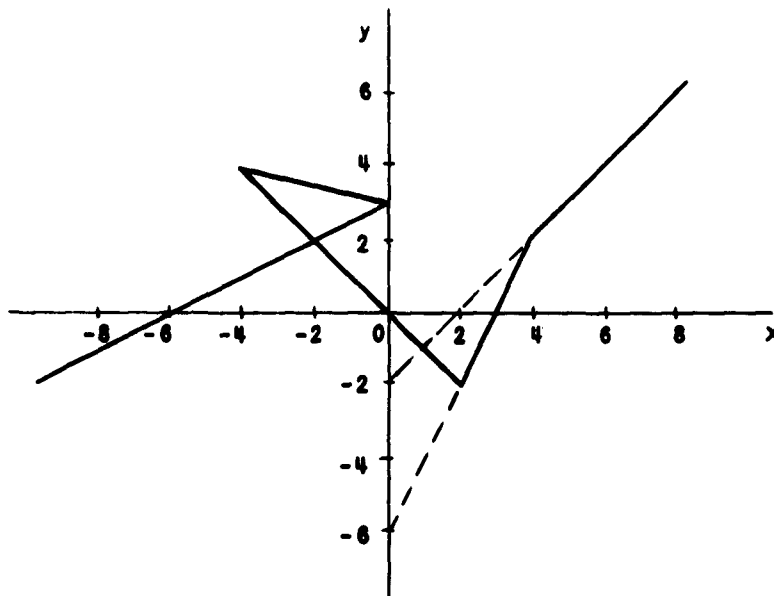
FIG. 12.   SELF-INTERSECTING PWL CURVE.

When a PWL operator operates on a constant, the result may be a single numerical value or a set of values, depending on whether the PWL curve is single- or multi-valued at the point in question. For B defined by Eq. (6), $B(5) = 3$, but $B(-1) = 1$, $5/2$, or $13/4$.

The use of PWL operators to represent curves that are defined for a semi-infinite range of x requires a slight modification in the notation. A curve that is defined only for $x \leq x_a$ starts on the left and ends on the left. A PWL operator in the previously defined form could be used to represent the curve, except that the last section applies to the range $x \leq b_{n-1}$ instead of $x \geq b_{n-1}$. To indicate this difference, a bar is placed on the right bracket of the PWL operator opposite the last intercept and slope. For example, Fig. 7f is represented by

- 18 -

$$v = \begin{pmatrix} -1_a, & 1 \\ 1_a, & -1 \end{pmatrix} \quad (1)$$

The convention will be established that, whenever a PWL curve both begins and ends on the left, the lower of the two end sections will be listed first in the PWL operator.

When a PWL curve is defined only for $x \geq x_a$, it starts on the right and ends on the right. To indicate that the first section applies to the range $x \geq b_1$ instead of $x \leq b_1$, a bar is placed on the right bracket of the PWL operator opposite the <u>first</u> intercept and slope. For this type of PWL curve, the lower of the two end sections will also be listed first in the PWL operator. For example, Fig. 7d is represented by

$$v = \begin{pmatrix} 1_a, & -1 \\ -1_a, & 1 \end{pmatrix} \quad (1)$$

When working with PWL curves that are defined for all values of x, it is sometimes convenient to list the sections in the PWL operator in reverse order, starting on the right and ending on the left. In this case, a bar is placed opposite both the first and last sections in the PWL operator. Thus Eq. (5) could be rewritten as

$$A = \begin{pmatrix} 0, & 1 \\ 8/3, & 1/3 \\ 2, & 1 \\ 5, & 2 \\ -1, & 1/2 \end{pmatrix}$$

To indicate that a PWL curve is defined for only a finite range of x and that the curve has a closed-loop form, a double parenthesis will be placed on the right side of the corresponding PWL operator. For example, Fig. 9d is represented by

$$v = \begin{pmatrix} -1, & 0 \\ -1, & 1 \\ 1, & 0 \\ 1, & 1 \end{pmatrix} \quad (1)$$

C. AN ALTERNATE FORM OF PWL OPERATORS

When PWL operators are manipulated on a digital computer, an alternate representation turns out to be convenient. Instead of specifying a PWL operator in terms of intercepts and slopes, it is possible to work with breakpoints. Listing the coordinates of the breakpoints in order will determine all of the sections of a PWL curve except the two end sections. To determine these end sections, additional information must be supplied. Specifying the slope of each end segment would be adequate but would lead to a PWL operator composed of n-1 breakpoints and two slopes. This unsymmetrical form would result in complications when deriving rules for algebraic operations with PWL operators. Instead of specifying the slope of an end section, the coordinates of a point on this section can be supplied. This procedure leads to the representation of an n-section PWL curve by a PWL operator of the form

$$
A = \begin{bmatrix}
x_0, & y_0 \\
x_1, & y_1 \\
\vdots & \\
x_k, & y_k \\
\vdots & \\
x_n, & y_n
\end{bmatrix}
\tag{7}
$$

where (x_k, y_k) are the coordinates of the kth breakpoint ($k = 1,2,\ldots,n-1$),

(x_0, y_0) are the coordinates of a point on the first section, and

(x_n, y_n) are the coordinates of a point on the last section.

The points on the end sections may be chosen arbitrarily, although for numerical work it is desirable to choose these points sufficiently far from the nearest breakpoints so that the slopes of the end sections can be determined accurately. Square brackets will be used to enclose the breakpoint form of PWL operator, which is defined above, to distinguish it from the slope-intercept form, which will be enclosed by parentheses. The breakpoint form has the disadvantage of requiring n+1 pairs of numbers to represent an n-section PWL curve compared with the slope-intercept form, which requires only n pairs. However, with the breakpoint form, it is unnecessary to use bars on the right bracket to indicate when a PWL curve starts on the right or ends on the left. If $x_0 < x_1$, the curve

starts on the left, and if $x_0 > x_1$, the curve starts on the right. If $x_{n-1} < x_n$, the curve ends on the right, and if $x_{n-1} > x_n$, the curve ends on the left.

Conversion from one form of PWL operator to the other is easy. Starting with the breakpoint form, Eq. (7), the slopes and intercepts for the other form, Eq. (4), are given by

$$r_k = \frac{y_k - y_{k-1}}{x_k - x_{k-1}} \tag{8}$$

$$q_k = y_k - r_k x_k \tag{9}$$

To convert the slope-intercept form to the breakpoint form, proceed as follows:

1. Calculate the b_k's by Eq. (3).
2. Choose $x_0 < b_1$ ($> b_1$ if the PWL curve starts on the right).
3. Set $x_k = b_k$ (for $k = 1,2,...,n-1$).
4. Choose $x_n > b_{n-1}$ ($< b_{n-1}$ if the PWL curve ends on the left).
5. Set $y_0 = q_1 + r_1 x_0$.
6. Set $y_k = q_k + r_k x_k$ (for $k = 1,2,...,n$).

Written in breakpoint form, Eqs. (5) and (6) become

$$A = \begin{bmatrix} -6, -4 \\ -4, -3 \\ -3, -1 \\ 1, 3 \\ 4, 4 \\ 6, 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -6, 0 \\ 0, 3 \\ -4, 4 \\ 2, -2 \\ 4, 2 \\ 6, 4 \end{bmatrix}$$

These PWL operators can easily be written down by inspection of Figs. 5b and 12.
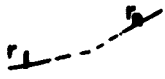
If A operating on x yields a single value for every value of x, A will be called a single-valued PWL operator. If A(x) represents a monotonic function of x, A will be called a monotonic PWL operator. If A is a single-valued PWL operator in breakpoint form,

$$A(x) = \begin{bmatrix} x_o, & y_o \\ \vdots & \\ x_n, & y_n \end{bmatrix}(x) = \begin{cases} y_j & \text{if } x = x_j \\ y_j + (x - x_j)r_j & \text{if } x_{j-1} < x < x_j \\ y_o + (x - x_o)r_1 & \text{if } x < x_o \\ y_n + (x - x_n)r_n & \text{if } x > x_n \end{cases} \qquad (10)$$

where $\quad r_j = \dfrac{y_j - y_{j-1}}{x_j - x_{j-1}}$

The various forms of PWL operators that have been defined are summarized in Table 1. The type designation indicates whether the PWL curves start at the left or right and whether they end at the left or right, e.g., type LR starts at the left and ends at the right. It is unnecessary to use bars on the brackets of the breakpoint form to distinguish the different types because there can never be any ambiguity as to the directions of the end segments.

TABLE 1. TYPES OF PWL OPERATORS

PWL Operator Type Designation and Form of Curve	Form of Operator	
	Slope-Intercept	Breakpoint
LR	$\begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix}$	$\begin{bmatrix} x_o, & y_o \\ \vdots & \\ x_n, & y_n \end{bmatrix}$ $(x_o < x_1, \ x_{n-1} < x_n)$
LL	$\begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix}$	$\begin{bmatrix} x_o, & y_o \\ \vdots & \\ x_n, & y_n \end{bmatrix}$ $(x_o < x_1, \ x_{n-1} > x_n)$
RR	$\begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix}$	$\begin{bmatrix} x_o, & y_o \\ \vdots & \\ x_n, & y_n \end{bmatrix}$ $(x_o > x_1, \ x_{n-1} < x_n)$
RL	$\begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix}$	$\begin{bmatrix} x_o, & y_o \\ \vdots & \\ x_n, & y_n \end{bmatrix}$ $(x_o > x_1, \ x_{n-1} > x_n)$
Closed loop	$\begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix}$	$\begin{bmatrix} x_o, & y_o \\ \vdots & \\ x_n, & y_n \end{bmatrix}$ $(x_o = x_n, \ y_o = y_n)$

D. REPRESENTATION OF SOURCES AND DIODES BY PWL OPERATORS

When a section of a PWL curve has infinite slope, there is no difficulty representing the curve in terms of PWL operators in the breakpoint form, but some trouble arises when we try to represent it in slope-intercept form. We can get around this difficulty by assuming that the slope is very large, but still finite, and later taking the limit as the slope becomes infinite.

The v-i characteristic of an ideal current source has an infinite slope as shown in Fig. 13. If the current source has a very large shunt resistance L, then the characteristic intersects the v-axis at $v = -i_0 L$. In terms of PWL operators, the voltage across the current source is

$$v = (-i_0 L, L) \quad (i)$$

If we let $L \to \infty$, we can express the characteristic of the ideal current source symbolically as

$$v = (-i_0 \infty, \infty) \quad (i) \tag{11}$$

Similarly, for an ideal voltage source of value e, we will write

$$i = \lim_{L \to \infty} (-eL, L) \quad (v) = (-e\infty, \infty) \quad (v) \tag{12}$$
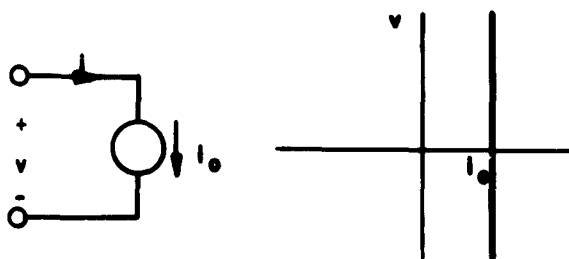
FIG. 13. V-I CHARACTERISTIC OF IDEAL CURRENT SOURCE.

The voltage across the source is

$$v = e + 0i = (e, 0)(i) \tag{13}$$

In terms of PWL operators, the v-i characteristic of a diode with forward resistance ϵ and reverse resistance L is

$$v = \begin{pmatrix} 0, & L \\ 0, & \epsilon \end{pmatrix}(i) \tag{ }$$

By letting $\epsilon \to 0$ and $L \to \infty$, we can express the ideal diode characteristic of Fig. 3 symbolically as

$$v = \begin{pmatrix} 0, & \infty \\ 0, & 0 \end{pmatrix}(i) \tag{14}$$

The series diode network of Fig. 14 and its dual, the parallel diode network of Fig. 15, occur frequently as building blocks in resistive diode networks. In terms of PWL operators, the v-i characteristics of these networks are respectively

$$v = \lim_{L \to \infty} \begin{pmatrix} e, & L \\ e, & r \end{pmatrix}(i) = \begin{pmatrix} e, & \infty \\ e, & r \end{pmatrix}(i) \tag{15}$$

$$v = \lim_{\epsilon \to 0} \begin{pmatrix} -i_o, & \epsilon \\ -i_o g, & g \end{pmatrix}(i) = \begin{pmatrix} 0, & 0 \\ -i_o g, & g \end{pmatrix}(i) \tag{16}$$

By considering the two states of the diode, the PWL operator that characterizes a network of either type can be written down by inspection of the network. When solving some types of PWL network problems, it is convenient to work in terms of ϵ and L, and then let $\epsilon \to 0$ and $L \to \infty$ after the solution has been obtained.

E. STERN'S METHOD FOR REPRESENTATION OF PWL FUNCTIONS

Stern [Refs. 4,5] represents single-valued PWL functions in terms of transformations ϕ^+ and ϕ^-, which have the following properties:

$$(x_1, x_2, \ldots, x_n)\phi^+ = \text{maximum value of } x_1, x_2, \ldots, \text{ and } x_n \tag{17}$$

$$(x_1, x_2, \ldots, x_n)\phi^- = \text{minimum value of } x_1, x_2, \ldots, \text{ and } x_n \tag{18}$$

Using these transformations, the PWL function of Fig. 5b is

$$v = [-1 + 1/2, 1, (5 + 2i, 2 + i, 8/3 + 1/3)\phi^-]\phi^+ \tag{19}$$
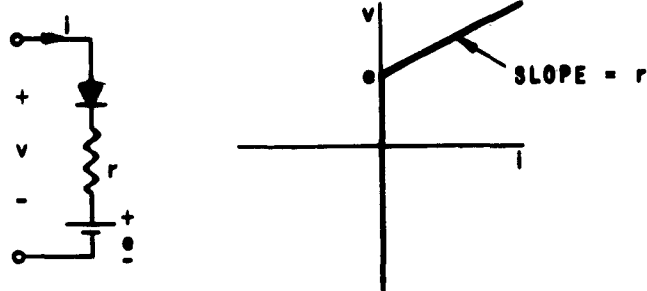
FIG. 14. SERIES DIODE NETWORK AND V-I CHARACTERISTIC.
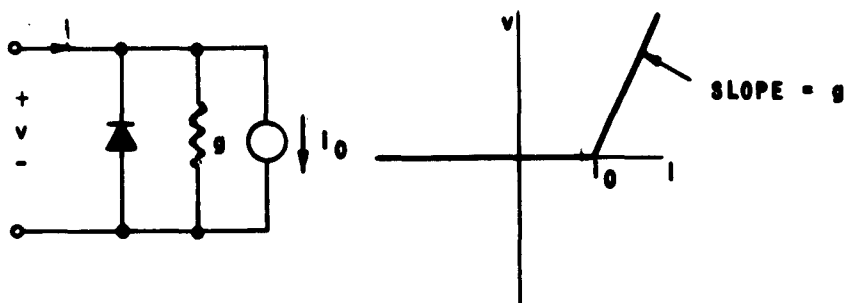
FIG. 15. PARALLEL DIODE NETWORK AND V-I CHARACTERISTIC.

Any single-valued PWL function of x can be represented by a generalization of one of the following forms:

$$y = ((...((x_{11},x_{12},...,x_{1a})\phi^+,x_{21},x_{22},...,x_{2b})\phi^+,...)\phi^-,x_{n1},x_{n2},...,x_{nm})\phi^-$$

or   (20)

$$y = [(x_{11},x_{12},...,x_{1a})\phi^-,(x_{21},x_{22},...,x_{2b})\phi^+,...,(x_{n1},x_{n2},...,x_{nm})\phi^-]\phi^+$$

(20a)

where x_{1j} is a linear function of x. Since the transformations ϕ^+ and ϕ^- yield a single value, multi-valued curves cannot be expressed in this form. The form and complexity of a PWL function expressed in terms of ϕ^+ and ϕ^- generally depend on the shape of the PWL curve as well as on the number of sections. For example, if we change the last section in Fig. 5b from v = 1 to v = 2 + 1/2, Eq. (19) must be replaced by

$$v = \left\{ [-1 + 1/2, (5 + 21, 2 + 1)\phi^-]\phi^+, (8/3 + 1/3, 2 + 1/2)\phi^+ \right\}\phi^-$$

(21)

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Any PWL curve which can be represented in Stern's notation can be represented in terms of PWL operators, but not conversely. ϕ^{+-} transformations that contain two arguments can be converted to PWL operators by the following relations:

$$(x,y)\phi^{+} = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} (x - y) + y = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} (y - x) + x \qquad (22)$$

$$(x,y)\phi^{-} = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} (x - y) + y = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} (y - x) + x \qquad (23)$$

These relations can readily be verified by considering the two possible cases, $x > y$ and $x < y$. For example, if $x > y$, Eq. 22 becomes

$$x = 1 \cdot (x - y) + y = 0 \cdot (y - x) + x$$

A ϕ^{+-} transformation that contains more than two arguments can first be reduced to a series of ϕ^{+-} transformations, each of which has only two arguments, by using the relationship

$$(x_1, x_2, x_3, \ldots, x_n)\phi^{+-} = ((\ldots((x_1, x_2)\phi^{+-}, x_3)\phi^{+-}, \ldots)\phi^{+-}, x_n)\phi^{+-} \qquad (24)$$

Since it is possible to represent multi-valued PWL curves by PWL operators but not by ϕ^{+-} transformations, it is not always possible to convert a PWL operator to an equivalent expression involving ϕ^{+-} transformations.

An algebraic method for analyzing PWL networks, based on Stern's notation, is discussed in Appendix A. The relative advantages and disadvantages of Stern's methods and PWL-operator methods are discussed in Section X. B. A representation for PWL curves in terms of absolute values is discussed in Appendix B.

III. ALGEBRAIC OPERATIONS WITH PWL OPERATORS

PWL operators have been defined to represent the characteristics of PWL elements in a compact form. The next step in formulating a systematic analysis of PWL networks is to define algebraic operations with PWL operators. Inversion, addition, multiplication, and other operations will be defined for PWL operators in both slope-intercept and breakpoint forms. The algebraic properties of these operations will be examined.

A. INVERSION OF PWL OPERATORS

Given the PWL operator which represents the v-i characteristic of a PWL resistor, a method is needed for determining the PWL operator that represents the corresponding i-v characteristic. In order to solve an equation of the form $y = A(x)$ for x, we will define the inverse of the PWL operator, A^{-1}, so that $x = A^{-1}(y)$. Fig. 16 shows a PWL curve and its inverse. Graphically, finding the inverse of a PWL curve amounts to interchanging the x and y axes by reflecting the curve about a 45° line drawn through the origin.

1. Inversion in Slope-Intercept Form

For a PWL curve that consists of a single section,

$$y = q_k + r_k x = (q_k, r_k)\,(x)$$

Solving this linear equation for x yields

$$x = -\frac{q_k}{r_k} + \frac{1}{r_k}\,y = \left(-\frac{q_k}{r_k} \,,\, \frac{1}{r_k} \right)(y)$$

In terms of PWL operators,

$$(q_k, r_k)^{-1} = \left(-\frac{q_k}{r_k} \,,\, \frac{1}{r_k} \right) \tag{25}$$

To generalize this inversion procedure to an nth-order PWL operator, Eq. (25) is applied to each section. If

$$A = \begin{pmatrix} q_1, r_1 \\ q_2, r_2 \\ \vdots \\ q_n, r_n \end{pmatrix}$$

represents a PWL curve which starts in the third quadrant and ends in the first quadrant, then

$$A^{-1} = \begin{pmatrix} -q_1/r_1, \ 1/r_1 \\ -q_2/r_2, \ 1/r_2 \\ \vdots \\ -q_n/r_n, \ 1/r_n \end{pmatrix} \tag{26}$$

Since the intercept and slope of each section is correct and the sections are listed in the correct order, Eq. (26) must be the correct representation of the inverse. For Fig. 16,
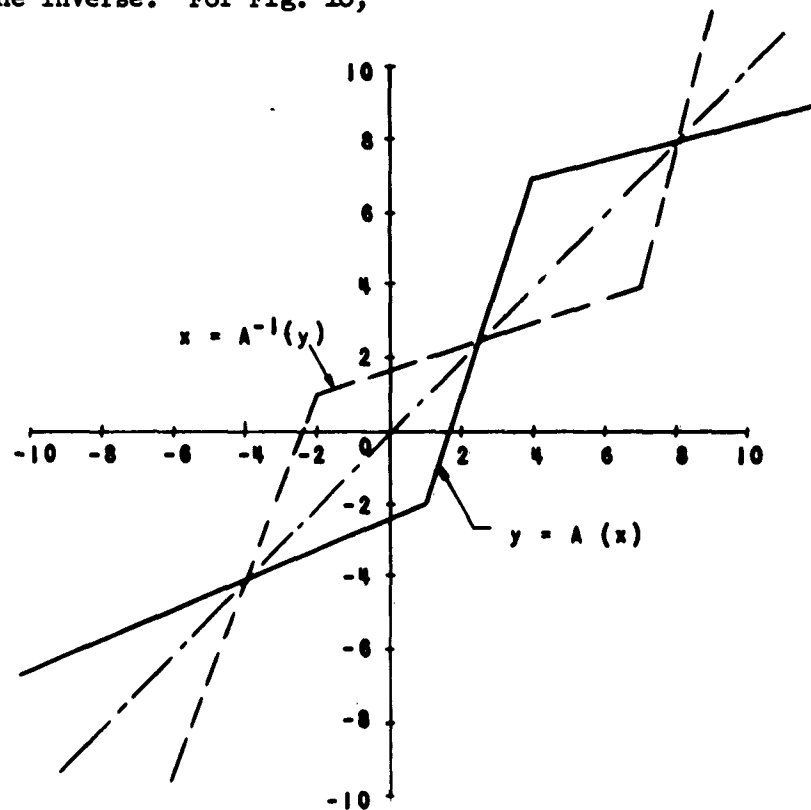
FIG. 16. A PWL CURVE AND ITS INVERSE.

$$A = \begin{pmatrix} -12/5, & 2/5 \\ -5, & 3 \\ 6, & 1/4 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 6, & 5/2 \\ 5/3, & 1/3 \\ -24, & 4 \end{pmatrix}$$

If a PWL curve starts or ends in the first or third quadrant, the inverse curve starts or ends in the <u>same</u> quadrant as the original curve; however, if the curve starts or ends in the second or fourth quadrant, the inverse curve starts or ends in the <u>opposite</u> quadrant. When a PWL curve starts or ends in the second or fourth quadrant, the initial or final slope is negative, and the corresponding PWL operator has $r_1 < 0$ or $r_n < 0$. In this case, the inverse is still formed as in Eq.(26), but bars must be added to (or omitted from) the right bracket of the inverse PWL operator according to the following rules:

If $r_1 < 0$, add (or omit)[*] a bar opposite the first section.

If $r_n < 0$, add (or omit)[*] a bar opposite the last section.

Examples of the application of these rules are

$$\begin{pmatrix} 6, & 2 \\ 0, & 1/2 \\ 1/2, & -1/2 \end{pmatrix}^{-1} = \begin{pmatrix} -3, & 1/2 \\ 0, & 2 \\ 1, & -2 \,\rceil \end{pmatrix}, \quad \begin{pmatrix} 0, & -1/2 \\ 0, & 1/2 \\ -2, & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0, & -2 \,\rceil \\ 0, & 2 \\ 2, & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1, & -1/2 \\ -1, & -1 \\ -2, & -1/3 \end{pmatrix}^{-1} = \begin{pmatrix} 2, & -2 \,\rceil \\ -1, & -1 \\ -3/2, & -3 \,\rceil \end{pmatrix} = \begin{pmatrix} -3/2, & -3 \\ -1, & -1 \\ 2, & -2 \end{pmatrix}$$

When PWL curves are classified according to the quadrants in which the curves begin and end, there are 10 possible types (Fig. 17). By reflecting these curves about a 45° line, it is seen that curves of types (a), (b), (g), and (h) have inverses of the same type as the original curve, but curves of types (c) and (d), (e) and (f), and (i) and (j) are inverses of each other.

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[*] Omit the bar from the inverse if the original PWL operator has a bar in the corresponding position.
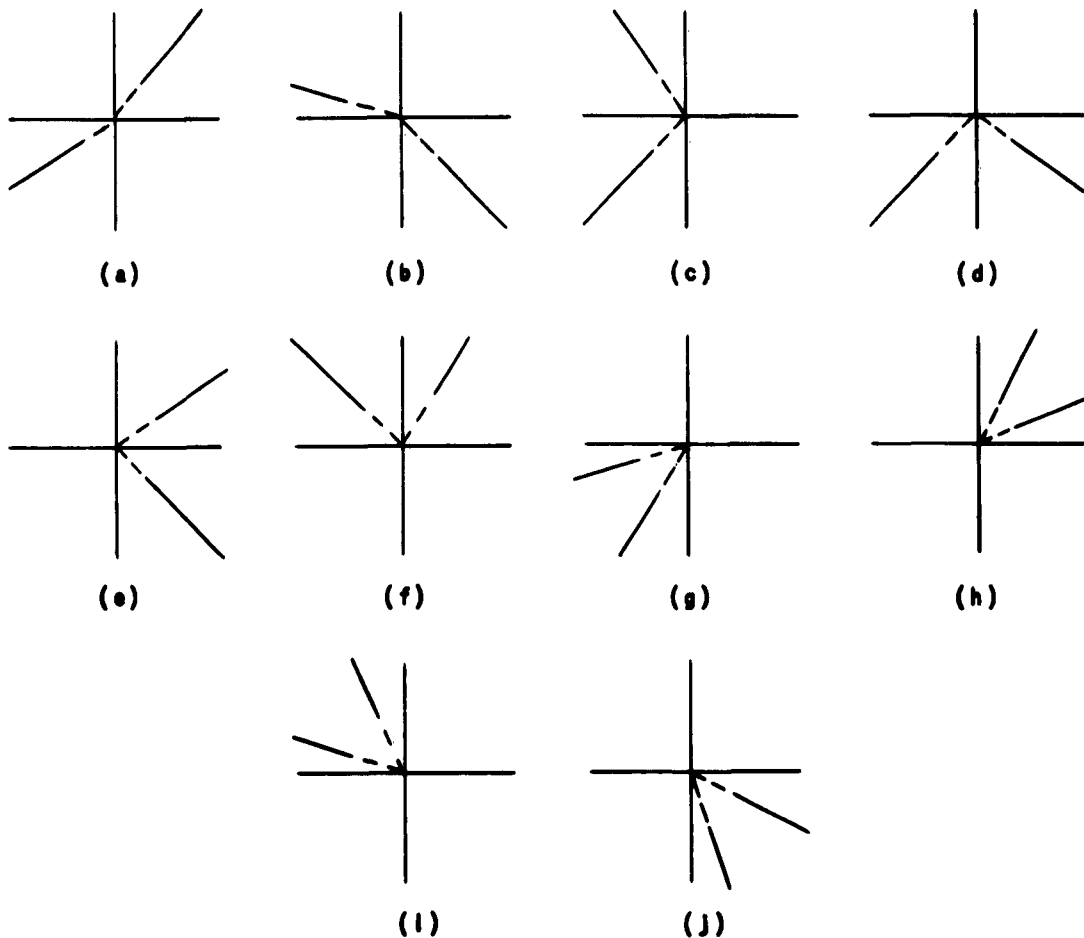
FIG. 17. INITIAL AND FINAL SEGMENTS FOR TEN TYPES OF PWL CURVES.

2. Inversion in Breakpoint Form

The inverse of a PWL operator in breakpoint form is easy to find. Since finding the inverse of a PWL curve amounts to interchanging the x- and y-coordinates of every point, interchanging the x- and y-coordinates of the breakpoints will give the coordinates of the breakpoints of the inverse. Therefore,

$$\begin{bmatrix} x_0, y_0 \\ \vdots \\ x_n, y_n \end{bmatrix}^{-1} = \begin{bmatrix} y_0, x_0 \\ \vdots \\ y_n, x_n \end{bmatrix} \tag{27}$$

For the curve of Fig. 16,

$$A = \begin{bmatrix} -9, & -6 \\ 1, & -2 \\ 4, & 7 \\ 10, 8.5 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -6, & -9 \\ -2, & 1 \\ 7, & 4 \\ 8.5, & 10 \end{bmatrix}$$

The inverse of a PWL curve can always be found graphically by reflecting the curve about a 45° line. The inverse of a PWL operator in breakpoint form can always be found by Eq.(27). The inverse of a PWL operator in slope-intercept form can always be found by Eq.(26), provided that all of the slopes are non-zero. If any slope is zero, its reciprocal is infinite, and a special method is needed to express the inverse in slope-intercept form. If a section has zero slope, we can replace 0 with ϵ and take the limit as $\epsilon \to 0$ after the inversion has been carried out. Symbolically, we can write

$$(q,0)^{-1} = \lim_{\epsilon \to 0} (q,\epsilon)^{-1} = \lim_{\epsilon \to 0} (\tfrac{-q}{\epsilon}, \tfrac{1}{\epsilon}) = \lim_{L \to \infty} (-qL, L) = (-q\infty, \infty).$$

B. ADDITION OF PWL OPERATORS

When two PWL resistors in series carry a current i, the voltage across the combination is

$$v = v_1 + v_2 = R_1 (i) + R_2 (i)$$

where R_1 and R_2 are the PWL operators that represent the characteristics of the two resistors. We will define addition of PWL operators so that we can write

$$v = R_1 (i) + R_2 (i) = (R_1 + R_2) (i)$$

Addition will first be defined for single-valued PWL operators, and later the definition will be extended to multi-valued operators. The sum of two single-valued PWL operators is defined so that

$$A(x) + B(x) = (A + B) (x) \tag{28}$$

for all values of x.

Graphical addition of PWL curves is illustrated in Fig. 18. Each value of y on the sum curve is obtained by adding the values of y on the curves being added which correspond to the same value of x.
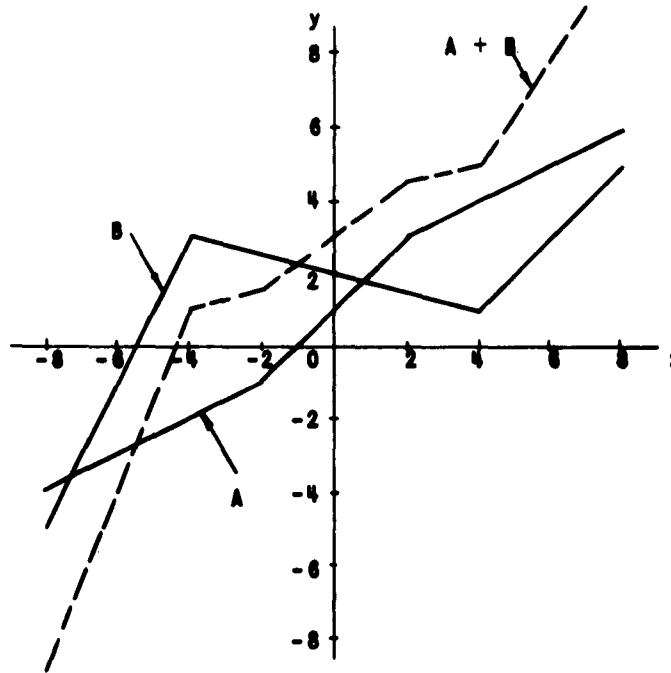
FIG. 18. ADDITION OF PWL CURVES.

Since a PWL curve is determined by its breakpoints, it is necessary only to carry out the addition for values of x where either curve has a break-point and for one point on each end segment.

1. Addition in Slope-Intercept Form

For the case where each PWL curve has only a single section with

$$y_1 = q_j + r_j x = (q_j, \; r_j) \, (x)$$

$$y_2 = q_k + r_k x = (q_k, \; r_k) \, (x)$$

the sum is

$$y_1 + y_2 = (q_j + q_k) + (r_j + r_k) \, x = (q_j + q_k, \; r_j + r_k) \, (x)$$

In terms of PWL operators,

$$(q_j, \; r_j) + (q_k, \; r_k) = (q_j + q_k, \; r_j + r_k) \qquad (29)$$

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In words, Eq. (29) can be stated as the intercept of the sum is the sum of the intercepts, and the slope of the sum is the sum of the slopes. The sum of a first-order PWL operator and an nth-order PWL operator can be found by applying Eq. (29) n times:

$$(q_o, r_o) \ + \ \begin{pmatrix} q_1, r_1 \\ \vdots \\ q_n, r_n \end{pmatrix} \ = \ \begin{pmatrix} q_o+q_1, r_o+r_1 \\ \vdots \\ q_o+q_n, r_o+r_n \end{pmatrix} \quad (29a)$$

To generalize the addition procedure to higher-order PWL operators, pairs of sections are selected according to the relative order of the breakpoints, and each pair is added by Eq. (29). After a pair of sections has been added, the breakpoints that follow those sections are compared, and the lesser breakpoint is selected. If the breakpoint of A has been chosen, we move down[*] to the next section of A, or if the breakpoint of B has been chosen, we move down to the next section of B. For Fig. 18,

$$A = \begin{pmatrix} 0,1/2 \\ 1, 1 \\ 2,1/2 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 11, 2 \\ 2,-1/4 \\ -3, 1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 0,1/2 \\ 1, 1 \\ 2,1/2 \end{pmatrix} \begin{matrix} -2 \\ \\ 2 \end{matrix} + \begin{pmatrix} 11, 2 \\ 2,-1/4 \\ -3, 1 \end{pmatrix} \begin{matrix} -4 \\ \\ 4 \end{matrix} = \begin{pmatrix} 11,5/2 \\ 2,1/4 \\ 3,3/4 \\ 4,1/4 \\ -1,3/2 \end{pmatrix} \quad (30)$$

To find the sum, the breakpoints of A and B are first calculated by Eq. (3) and listed beside the PWL operators. The first sections of both operators are added to get the first section of the sum. Since $-4 < -2$, we move to the next section of B and add $(0,\frac{1}{2})$ and $(2,-\frac{1}{4})$ to get $(2,\frac{1}{4})$. Next, since $-2 < 4$, we add the next section of A to the same section of B, which gives $(1,1) + (2,-\frac{1}{4}) = (3,3/4)$. This process is continued until finally we add the two last sections to get the last section of the sum.

---

[*] Moving down the column in a single-valued PWL operator corresponds to moving to the right along the corresponding PWL curve.

The rule for addition of single-valued PWL operators in slope-intercept form can be stated formally as follows. The sum of two PWL operators

$$A = \begin{pmatrix} q_{11}, r_{11} \\ \vdots \\ q_{1m}, r_{1m} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} q_{21}, r_{21} \\ \vdots \\ q_{2n}, r_{2n} \end{pmatrix}$$

is a PWL operator of the form

$$C = A + B = \begin{pmatrix} q_{31}, r_{31} \\ \vdots \\ q_{3\ell}, r_{3\ell} \end{pmatrix} \qquad\qquad (31)$$

where $q_{3i} = q_{1j} + q_{2k}$ and $r_{3i} = r_{1j} + r_{2k}$ with j and k determined as follows:

1. For $i = 1$, $j = k = 1$
2. If for $i = i'$, $j = j'$ and $k = k'$, then for $i = i' + 1$,

   $j = j' + 1$, $k = k'$ if $b_{1j'} < b_{2k'}$

   $j = j' + 1$, $k = k' + 1$ if $b_{1j'} = b_{2k'}$

   $j = j'$, $k = k' + 1$ if $b_{1j'} > b_{2k'}$

3. For the last section ($i = \ell$), $j = m$ and $k = n$.

The breakpoints in the above rule are given by

$$b_{1j'} = \frac{q_{1,j'+1} - q_{1j'}}{r_{1j'} - r_{1,j'+1}} \qquad (j' = 1,2,\ldots,m-1), \qquad b_{1m} = \infty$$

$$b_{2k'} = \frac{q_{2,k'+1} - q_{2k'}}{r_{2k'} - r_{2,k'+1}} \qquad (k' = 1,2,\ldots,n-1), \qquad b_{2n} = \infty$$

When two successive sections of C have the same intercepts and the same slopes, both sections represent the same straight line, and one of the sections should be deleted since it is redundant. Note that m does not have to equal n, and ℓ will generally be greater than either m or n.

2. Addition in Breakpoint Form

Addition of single-valued PWL operators in breakpoint form can be expressed by the equation

$$C = A + B = \begin{bmatrix} x_{10}, \ y_{10} \\ \vdots \\ x_{1m}, \ y_{1m} \end{bmatrix} + \begin{bmatrix} x_{20}, \ y_{20} \\ \vdots \\ x_{2n}, \ y_{2n} \end{bmatrix} = \begin{bmatrix} x_{30}, \ y_{30} \\ \vdots \\ x_{3\ell}, \ y_{3\ell} \end{bmatrix} \tag{32}$$

The x-column of C is formed as follows:

1. x_{30} = minimum of x_{10} and x_{20}

2. To obtain $x_{31}, \ldots, x_{3,\ell-1}$ arrange $x_{11}, \ldots, x_{1,m-1}$ and $x_{21}, \ldots, x_{2,n-1}$ in increasing order.

3. $x_{3\ell}$ = maximum of x_{1m} and x_{2n}.

The entries in the y-column of C are

$$y_{3i} = A(x_{3i}) + B(x_{3i}) \tag{32a}$$

where $A(x_{3i})$ and $B(x_{3i})$ are evaluated by interpolation between the breakpoints using Eq. (10). For Fig. 18,

$$A + B = \begin{bmatrix} -4, -2 \\ -2, -1 \\ 2, \ 3 \\ 4, \ 4 \end{bmatrix} \begin{matrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{matrix} + \begin{bmatrix} -6, -1 \\ -4, \ 3 \\ 4, \ 1 \\ 6, \ 3 \end{bmatrix} \begin{matrix} 2 \\ -1 \\ 1 \end{matrix} = \begin{bmatrix} -6, -4 \\ -4, \ 1 \\ -2, 3/2 \\ 2, 9/2 \\ 4, \ 5 \\ 6, \ 8 \end{bmatrix}$$

For convenience in interpolation, the slopes of the sections can be listed beside A and B as shown above. A typical calculation for determination of y_{3i} is

$$y_{32} = A(-2) + B(-2) = -1 + 3 + [-2-(-4)](-\tfrac{1}{4}) = 3/2$$

During the addition process, redundant breakpoints may be introduced. If

$$(x_{j+1} - x_{j-1})(y_j - y_{j-1}) = (x_j - x_{j-1})(y_{j+1} - y_{j-1}) \tag{33}$$

the point (x_j, y_j) lies on the line joining (x_{j-1}, y_{j-1}) and (x_{j+1}, y_{j+1}) and therefore may be deleted.

C. MULTIPLICATION OF PWL OPERATORS

Inversion and addition of PWL operators have been defined, and now a multiplication operation will be introduced. Consider two PWL networks connected in cascade (Fig. 19), and assume that the loading effect of the second network on the first is negligible. If the transfer characteristics of the networks are represented by PWL operators with $v_2 = B(v_1)$ and $v_3 = A(v_2)$, we would like to define multiplication of PWL operators so that we can substitute $B(v_1)$ for v_2 and write $v_3 = AB(v_1)$. Then the product AB would represent the overall transfer characteristic of the two networks in cascade. In general, multiplication of PWL operators will be defined so that if $z = A(y)$ and $y = B(x)$, then $z = AB(x)$. Note that this is not ordinary multiplication, but is a substitution type of operation.

Graphical multiplication of PWL curves is illustrated in Fig. 20. Starting with plots of z vs y and y vs x, the object of the multiplication process is to eliminate y and obtain a plot of z vs x. When forming the product AB, B^{-1} is plotted below A so that the common variable, y, can be measured on the same horizontal scale for both curves. Since x is measured on a vertical scale for B^{-1} and on a horizontal scale for AB, the two scales for x can be related by reflection in a 45° line. For a given value of y, the corresponding values of z and x can be determined graphically and plotted on the product curve. The graphical construction required to do this is shown for typical points. Each point on the AB curve is determined as the intersection of two dashed lines. Since the breakpoints determine the PWL curve, this procedure need be carried out only at the breakpoints and at one point on each end segment.

FIG. 19. CASCADE CONNECTION OF PWL NETWORKS.
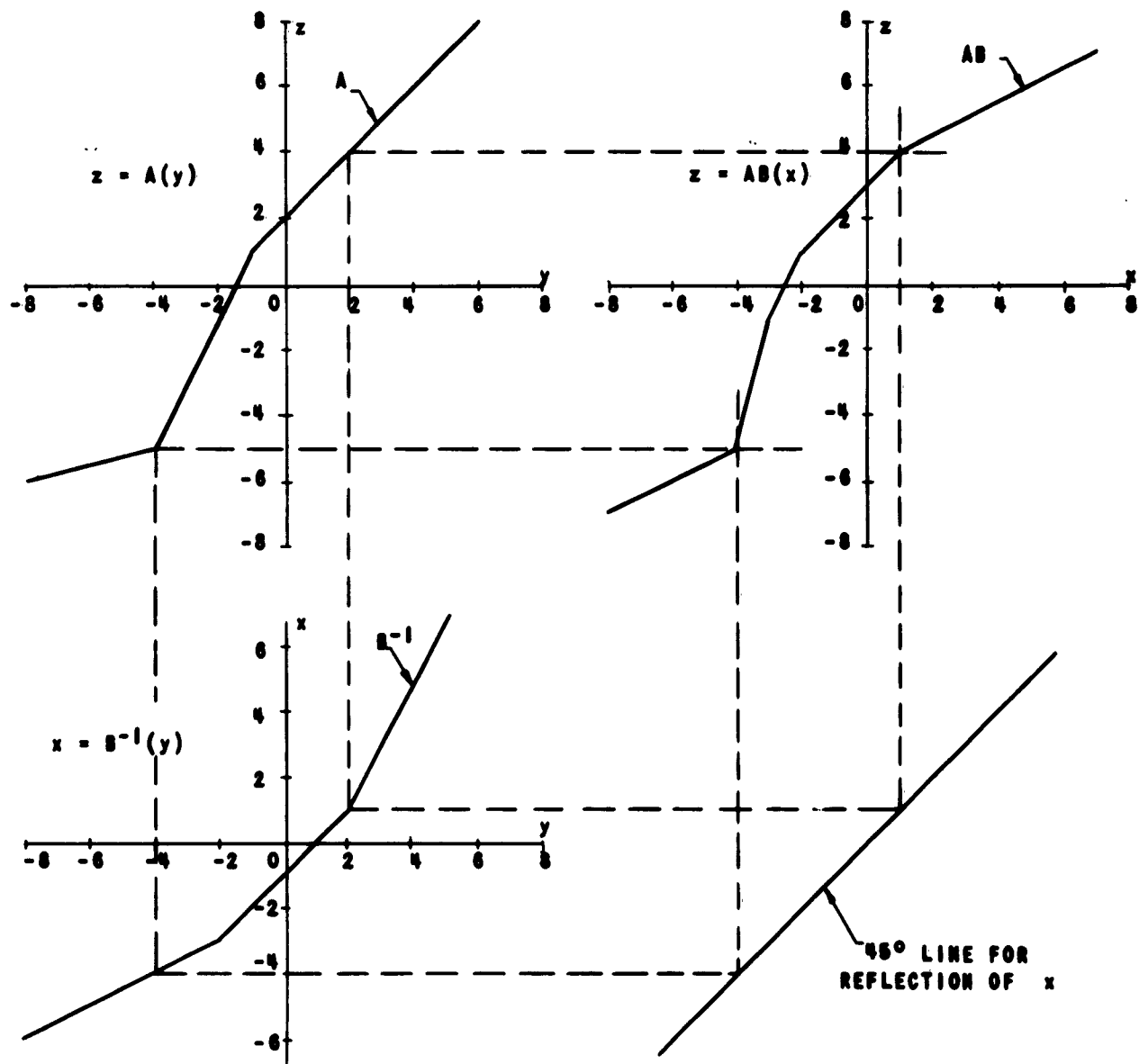
FIG. 20. GRAPHICAL MULTIPLICATION OF PWL CURVES.

1. Multiplication in Slope-Intercept Form

The product of two PWL operators, AB, will first be defined for the case where A and B^{-1} are single-valued, and later the definition will be extended to the multi-valued case. For the case where each PWL operator has only a single section with

$$z = q_j + r_j y = (q_j, r_j)\,(y)$$

$$y = q_k + r_k x = (q_k, r_k)\,(x)$$

substituting the second equation into the first yields

$$z = (q_j + r_j q_k) + (r_j r_k)x = (q_j + r_j q_k,\ r_j r_k)\,(x) = (q_j, r_j)\,(q_k, r_k)\,(x)$$

In terms of PWL operators,

$$(q_j, r_j)\,(q_k, r_k) = (q_j + r_j q_k,\ r_j r_k) \tag{34}$$

To multiply a first order PWL operator by an nth order operator, Eq.(34) is applied n times:

$$(q_o,\ r_o)\begin{pmatrix} q_1,\ r_1 \\ \vdots \\ q_n,\ r_n \end{pmatrix} = \begin{pmatrix} q_o + r_o q_1,\ r_o r_1 \\ \vdots \\ q_o + r_o q_n,\ r_o r_n \end{pmatrix} \tag{35}$$

The product of an nth-order PWL operator times a first-order operator is obtained in a similar manner:

$$\begin{pmatrix} q_1,\ r_1 \\ \vdots \\ q_n,\ r_n \end{pmatrix}(q_o,\ r_o) = \begin{pmatrix} q_1 + r_1 q_o,\ r_1 r_o \\ \vdots \\ q_n + r_n q_o,\ r_n r_o \end{pmatrix} \tag{36}$$

It may be necessary to add bars to the right bracket of the product if r_o, r_1, or r_n is negative.

The multiplication procedure can be generalized to higher-order PWL operators in a manner similar to the addition procedure. Pairs of

sections are selected according to the relative order of the breakpoints, and each pair is multiplied by using Eq. (34). From the graphical construction, it is clear that we must compare breakpoints of A and B^{-1} instead of A and B as was done in the addition procedure. The breakpoints of B^{-1}, i.e., the y-coordinates of the breakpoints of B, can be found by the relation

$$c_k = q_k + r_k b_k = q_k + r_k \left(\frac{q_{k+1} - q_k}{r_k - r_{k+1}} \right) = \frac{q_k r_{k+1} - q_{k+1} r_k}{r_{k+1} - r_k} \tag{37}$$

For Fig. 20,

$$A = \begin{pmatrix} -4,1/4 \\ 3, & 2 \\ 2, & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 4, & 2 \\ 1, & 1 \\ 3/2,1/2 \end{pmatrix}$$

$$AB = \begin{pmatrix} -4,1/4 \\ 3, & 2 \\ 2, & 1 \end{pmatrix} \begin{matrix} -4 \\ -1 \end{matrix} \begin{matrix} -2 \\ 2 \end{matrix} \begin{pmatrix} 4, & 2 \\ 1, & 1 \\ 3/2,1/2 \end{pmatrix} = \begin{pmatrix} -3,1/2 \\ 11, & 4 \\ 5, & 2 \\ 3, & 1 \\ 7/2,1/2 \end{pmatrix}$$

To find the product, the b_k's are calculated by Eq. (3) and listed beside A, and the c_k's are calculated by Eq. (37) and listed on the left of B. To get the first section of the product, the first sections of A and B are multiplied using Eq. (34). Since $-4 < -2$, we move to the next section of A and multiply (3,2) by (4,2) to get (11,4). Next, since $-2 < -1$, we multiply the same section of A by the next section of B to get (3,2) (1,1) = (5,2). This process is continued until finally we multiply the last sections of A and B to get the last section of AB.

The rule for multiplying a single-valued PWL operator by a PWL operator whose inverse is single-valued can now be stated. The product is given by

$$C = AB = \begin{pmatrix} q_{11}, r_{11} \\ \vdots \\ q_{1m}, r_{1m} \end{pmatrix} \begin{pmatrix} q_{21}, r_{21} \\ \vdots \\ q_{2n}, r_{2n} \end{pmatrix} = \begin{pmatrix} q_{31}, r_{31} \\ \vdots \\ q_{3\ell}, r_{3\ell} \end{pmatrix} \tag{38}$$

where $q_{3i} = q_{1j} + r_{1j} q_{2k}$ and $r_{3i} = r_{1j} r_{2k}$,

with j and k determined in the same manner as for the addition rule, Eq. (31), except that $b_{2k'}$ is replaced by

$$c_{k'} = \frac{q_{2k'}r_{2,k'+1} - q_{2,k'+1}r_{2k'}}{r_{2,k'+1} - r_{2k'}} \qquad (k' = 1,2,\ldots,n-1), \quad c_n = \infty$$

If B has a bar on the right bracket, C will have a bar in the corresponding position. For example,

$$\begin{pmatrix} 0, & 2 \\ 0, & -5 \end{pmatrix} \quad \begin{pmatrix} 0, & -.2 \\ 0, & 2 \end{pmatrix} = \begin{pmatrix} 0, & -.4 \\ 0, & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0, & 2 \\ 0, & .5 \end{pmatrix} \quad \begin{pmatrix} 0, & 1 \\ 0, & -5 \end{pmatrix} = \begin{pmatrix} 0, & 2 \\ 0, & -2.5 \end{pmatrix}$$

If B starts in the second quadrant and ends in the fourth, before carrying out the multiplication, the sections of B in Eq. (38) must be listed in reverse order so that the c_k's will increase with k. For example,

$$\begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} \begin{pmatrix} 1, & -1/2 \\ 0, & -1 \\ -1, & -1/2 \end{pmatrix} = \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} \begin{matrix} 0 \\ -2 \\ 2 \end{matrix} \begin{pmatrix} -1, & -1/2 \\ 0, & -1 \\ 1, & -1/2 \end{pmatrix} \begin{pmatrix} -1, & -1/2 \\ 0, & -1 \\ 0, & -1/2 \\ 1/2, & -1/4 \end{pmatrix} = \begin{pmatrix} 1/2, & -1/4 \\ 0, & -1/2 \\ 0, & -1 \\ -1, & -1/2 \end{pmatrix}$$

2. Multiplication in Breakpoint Form

If A and B^{-1} are single-valued PWL operators in breakpoint form, we can list the rows of A and B in an order such that the first column of A and the second column of B both increase as we move down the column. The product AB can then be expressed in the form

$$C = AB = \begin{bmatrix} x_{10}, & y_{10} \\ \vdots \\ x_{1m}, & y_{1m} \end{bmatrix} \begin{bmatrix} x_{20}, & y_{20} \\ \vdots \\ x_{2n}, & y_{2n} \end{bmatrix} = \begin{bmatrix} x_{30}, & y_{30} \\ \vdots \\ x_{3\ell}, & y_{3\ell} \end{bmatrix} \qquad (39)$$

To find C, an auxiliary Z-column is first constructed as follows:

1. z_1 = minimum of x_{10} and y_{20}.

2. To obtain z_2, z_3, $\ldots$, $z_{\ell-1}$, arrange x_{11}, x_{12}, $\ldots$, $x_{1,m-1}$ and

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y_{21}, y_{22}, ..., $y_{2,n-1}$ in increasing order.

3. z_ℓ = maximum of x_{1m} and x_{2n}.

Then to form C,

$$x_{3i} = B^{-1} (z_i) \quad \text{and} \quad y_{3i} = A(z_i) \quad (i = 0,1,2,\ldots,\ell) \qquad (39a)$$

are evaluated by interpolation between the breakpoints using Eq. (34).
For the example of Fig. 20,

$$A = \begin{bmatrix} -8,-6 \\ -4,-5 \\ -1, 1 \\ 2, 4 \end{bmatrix} \qquad B = \begin{bmatrix} -4,-4 \\ -3,-2 \\ 1, 2 \\ 3, 3 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} -4,-4 \\ -2,-3 \\ 2, 1 \\ 3, 3 \end{bmatrix}$$

$$Z = \begin{bmatrix} -8 \\ -4 \\ -2 \\ -1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} -6,-6 \\ -4,-5 \\ -3,-1 \\ -2, 1 \\ 1, 4 \\ 3, 5 \end{bmatrix}$$

A typical calculation for determining one of the rows of C is

$$x_{32} = B^{-1} (-2) = -3$$

$$x_{32} = A (-2) \quad = -5 + [-2 - (-4)]2 = -1$$

Extension of the multiplication procedure to multi-valued PWL operators
will be considered in Section III. J.

D. OPERATIONS WITH CONSTANTS

In the solution of networks that contain both linear and PWL elements,
it is frequently necessary to add a PWL operator to a constant or to mul-
tiply a PWL operator by a constant. In such cases, we will usually write

a + A, aA, and Aa as abbreviations for $(0,a) + A$, $(0,a)A$, and $A(0,a)$ respectively. When the PWL operator is in slope-intercept form, the following rules are special cases of Eqs. (29a), (35), and (36):

$$a + \begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix} = \begin{pmatrix} q_1, & r_1+a \\ \vdots & \\ q_n, & r_n+a \end{pmatrix} \qquad (40)$$

$$a \begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix} = \begin{pmatrix} aq_1, & ar_1 \\ \vdots & \\ aq_n, & ar_n \end{pmatrix} \qquad (41)$$

$$\begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix} a = \begin{cases} \begin{pmatrix} q_1, & ar_1 \\ \vdots & \\ q_n, & ar_n \end{pmatrix} & \text{if } a > 0 \\[2em] \begin{pmatrix} q_n, & ar_n \\ \vdots & \\ q_1, & ar_1 \end{pmatrix} & \text{if } a < 0 \end{cases} \qquad (42)$$

When the PWL operator is in breakpoint form, these rules become

$$a + \begin{bmatrix} x_0, & y_0 \\ \vdots & \\ x_n, & y_n \end{bmatrix} = \begin{bmatrix} x_0, & y_0 + ax_0 \\ \vdots & \\ x_n, & y_n + ax_n \end{bmatrix} \qquad (40a)$$

$$a \begin{bmatrix} x_0, & y_0 \\ \vdots & \\ x_n, & y_n \end{bmatrix} = \begin{bmatrix} x_0, & ay_0 \\ \vdots & \\ x_n, & ay_n \end{bmatrix} \qquad (41a)$$

$$\begin{bmatrix} x_0, & y_0 \\ \vdots & \\ x_n, & y_n \end{bmatrix} a = \begin{bmatrix} x_0/a, & y_0 \\ \vdots & \\ x_n/a, & y_n \end{bmatrix} \qquad (42a)$$

E. THE IDENTITY OPERATOR

The PWL operator, $(0,1)$, will be called the identity operator and given the special symbol I. The identity operator operating on anything gives back the same thing,

$$I(x) = (0,1)\,(x) = 0 + x = x$$

For every PWL operator, A,

$$AI = IA = A \qquad\qquad (43)$$

since for every section

$$(q_k, r_k)\,(0,1) = (0,1)\,(q_k, r_k) = (q_k, r_k)$$

If $y = A(x)$ is defined for all values of x,

$$x = A^{-1}\,(y) = A^{-1}A(x) = I(x),\ \text{so}\ A^{-1}A = I$$

If $x = A^{-1}(y)$ is defined for all values of y,

$$y = A(x) = AA^{-1}\,(y) = I(y),\ \text{so}\ AA^{-1} = I$$

It follows that if $y = A(x)$ is defined for all values of x and $x = A^{-1}(y)$ is defined for all values of y, then

$$AA^{-1} = A^{-1}A = I \qquad\qquad (44)$$

This equation is easily verified for a first-order PWL operator, $A = (q_k,\ r_k)$:

$$AA^{-1} = (q_k,\ r_k)\,(-q_k/r_k,\ 1/r_k) = (q_k - q_k,\ r_k/r_k) = (0,1) = I$$

$$A^{-1}A = (-q_k/r_k,\ 1/r_k)\,(q_k,\ r_k) = (-q_k/r_k + q_k/r_k,\ r_k/r_k) = (0,1) = I$$

For n^{th}-order PWL operators, the same result is obtained for every section so that the product of an operator and its inverse reduces to the identity.

F. NEGATION AND SUBTRACTION

The negative of a PWL operator will be defined so that if $y = A(x)$, then $-y = (-A)(x)$. The negative of a PWL curve is found graphically by interchanging y and $-y$ by reflecting the curve about the x-axis. This reflection is equivalent to changing the sign of every slope and intercept. Therefore, if

$$A = \begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix} \quad \text{then} \quad -A = (0,-1)A = \begin{pmatrix} -q_1, -r_1 \\ \vdots \\ -q_n, -r_n \end{pmatrix} \tag{45}$$

or, in breakpoint notation, if

$$A = \begin{bmatrix} x_0, & y_0 \\ \vdots & \\ x_n, & y_n \end{bmatrix} \quad \text{then} \quad -A = \begin{bmatrix} x_0, -y_0 \\ \vdots \\ x_n, -y_n \end{bmatrix} \tag{46}$$

Subtraction of PWL operators is then defined by

$$A - B = A + (-B) \tag{47}$$

i.e., to subtract B from A, add the negative of B to A.

It is useful to define a second type of negation with $A(-x) = \bar{A}(x)$. Graphically, this operation is performed by interchanging x and $-x$ by reflecting the PWL curve about the y-axis. Since $-x = (0,-1)(x)$, $A(-x) = A(0,-1)(x) = \bar{A}(x)$, so $\bar{A} = A(0,-1)$. If

$$A = \begin{pmatrix} q_1, & r_1 \\ \vdots & \\ q_n, & r_n \end{pmatrix} \quad \text{then} \quad \bar{A} = \begin{pmatrix} q_1, -r_1 \\ \vdots \\ q_n, -r_n \end{pmatrix} = \begin{pmatrix} q_n, -r_n \\ \vdots \\ q_1, -r_1 \end{pmatrix} \tag{48}$$

or, in breakpoint notation, if

$$A = \begin{bmatrix} x_0, & y_0 \\ \vdots & \\ x_n, & y_n \end{bmatrix} \quad \text{then} \quad \bar{A} = \begin{bmatrix} -x_n, & y_n \\ \vdots \\ -x_0, & y_0 \end{bmatrix} \tag{49}$$

The reordering of the sections is performed so that the sections will be listed in the customary order, i.e., from left to right.

Two useful relations which involve the two types of negation are

$$A(-B) = A(0,-1) B = \bar{A}B \tag{50}$$

and $(\bar{A})^{-1} = [A(0,-1)]^{-1} = (0,-1)A^{-1} = -(A^{-1})$ \hfill (50a)

G. ASSOCIATIVE, COMMUTATIVE, AND DISTRIBUTIVE LAWS

Both the associative and commutative laws are valid for addition of PWL operators. Since

$$[A(x) + B(x)] + C(x) = A(x) + [B(x) + C(x)] \qquad \text{for all } x,$$

it follows that

$$(A + B) + C = A + (B + C) \qquad \text{(associative law)}, \tag{51}$$

and since

$$A(x) + B(x) = B(x) + A(x) \qquad \text{for all } x,$$

it follows that

$$A + B = B + A \qquad \text{(commutative law)}. \tag{52}$$

For multiplication of PWL operators, the associative law is valid, but the commutative law is <u>not</u>. Since

$$A\left\{B[C(x)]\right\} = A[BC(x)] = AB[C(x)] \qquad \text{for all } x,$$

it follows that

$$A(BC) = (AB)C \qquad \text{(associative law)}. \tag{53}$$

The associative law can be verified directly for first-order PWL operators. If $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$,

$$A(BC) = (a_1, a_2)(b_1 + b_2 c_1, b_2 c_2) = (a_1 + a_2 b_1 + a_2 b_2 c_1, a_2 b_2 c_2)$$

and $(AB)C = (a_1 + a_2 b_1, a_2 b_2)(c_1, c_2) = (a_1 + a_2 b_1 + a_2 b_2 c_1, a_2 b_2 c_2)$

The commutative law does not hold in general for multiplication. Even for first-order PWL operators,

$$AB = (a_1, a_2)(b_1, b_2) = (a_1 + a_2 b_1, a_2 b_2)$$

but $$BA = (b_1, b_2)(a_1, a_2) = (b_1 + b_2 a_1, a_2 b_2)$$

and $$AB \neq BA.$$

The right distributive law is valid for PWL operators, but the left distributive law is not valid, i.e.,

$$(A + B)C = AC + BC \tag{54}$$

but $$A(B + C) \neq AB + AC \tag{55}$$

except in special cases. The validity of Eq. (54) follows from the way in which addition is defined. From the definition of addition,

$$(A + B)(y) = A(y) + B(y)$$

Now, substituting $C(x)$ for y,

$$(A + B)C(x) = AC(x) + BC(x) = (AC + BC)(x)$$

from which Eq. (54) follows. On the other hand, unless A is linear,

$$A(y_1 + y_2) \neq A(y_1) + A(y_2)$$

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If $y_1 = B(x)$ and $y_2 = C(x)$,

$$A[B(x) + C(x)] \neq AB(x) + AC(x)$$

and $\qquad\qquad A(B + C)(x) \neq (AB + AC)(x)$

which shows that the left distributive law is generally not true.

The right distributive law can be verified directly for first-order PWL operators. If $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$,

$$(A + B)C = (a_1{+}b_1,\ a_2{+}b_2)\,(c_1,c_2) = (a_1{+}b_1{+}a_2c_1{+}b_2c_1,\ a_2c_2{+}b_2c_2) \quad\text{and}$$

$$AC + BC = (a_1{+}a_2c_1,\ a_2c_2) + (b_1{+}b_2c_1,\ b_2c_2) = (a_1{+}b_1{+}a_2c_1{+}b_2c_1,\ a_2c_2{+}b_2c_2)$$

The left distributive law is generally not true even for first-order PWL operators:

$$A(B + C) = (a_1,a_2)\,(b_1{+}c_1, b_2{+}c_2) = (a_1{+}a_2b_1{+}a_2c_1, a_2b_2{+}a_2c_2) \quad\text{but}$$

$$AB + AC = (a_1{+}a_2b_1, a_2b_2) + (a_1{+}a_2c_1, a_2c_2) = (2a_1{+}a_2b_1{+}a_2c_2, a_2b_2{+}a_2c_2)$$

The only case in which the left distributive law holds is when A is a constant. If $A = (0,a)$,

$$(0,a)\,(B + C) = (0,a)B + (0,a)C$$

or $\qquad\qquad a(B + C) = aB + aC$ $\qquad\qquad\qquad$ (56)

This relation must be true, since premultiplying by a constant merely changes the vertical scale on the PWL curves.

A special left distributive law can be derived when A is of first-order. If $\quad A = (a_1, a_2)$, then

$$A(B + C) = (a_1, 0)\,(B + C) + (0, a_2)\,(B + C)$$
$$= (a_1, a_2)B + (0, a_2)C = AB + (0, a_2)C \qquad (57)$$

H. SHIFTING OF PWL OPERATORS

Given an expression of the form $y = A(x + b)$ where b is a constant, we would like to find a new PWL operator, A_b, such that $y = A_b(x)$. The PWL curve that A_b represents can be obtained by shifting the PWL curve for A a distance of b units to the left along the x-axis. For the kth section of A,

$$y = (q_k, r_k)(x + b) = q_k + r_k b + r_k x = (q_k + r_k b, r_k)(x) \qquad (58)$$

The <u>shifting</u> <u>rule</u> for an nth-order PWL operator is, therefore,

$$A(x + b) = \begin{pmatrix} q_1, r_1 \\ \vdots \\ q_n, r_n \end{pmatrix}(x + b) = \begin{pmatrix} q_1 + r_1 b, r_1 \\ \vdots \\ q_n + r_n b, r_n \end{pmatrix}(x) = A_b(x) \qquad (59)$$

The kth breakpoint of A_b is

$$\frac{(q_k + r_k b) - (q_{k+1} + r_{k+1} b)}{r_{k+1} - r_k} = \frac{q_k - q_{k+1}}{r_{k+1} - r_k} - b = b_k - b \qquad (60)$$

where b_k is the kth breakpoint of A, which indicates that all of the breakpoints have been shifted b units to the left.

The shifting rule can be applied even if b is a variable instead of a constant, but in this case the breakpoints of A_b are variable. Since addition or multiplication of PWL operators can be carried out only if the relative values of the breakpoints are known, it is usually not helpful to apply the shifting rule unless b is a known constant.

I. ADDITION OF MULTI-VALUED PWL OPERATORS

Addition of single-valued PWL operators has been previously defined. We now wish to extend the definition of addition to include multi-valued curves. First, the addition rule will be extended to the sum of a multi-valued and a single-valued PWL operator, and then finally to the sum of two multi-valued operators. In carrying out this extension, we want to

make sure that the sum of two PWL operators which represent the characteristics of PWL resistors correctly represents the characteristic of the two resistors in series.

Some of the procedures that will be developed for addition and multiplication of multi-valued PWL operators are fairly complicated to carry out by hand; however, these procedures have been programmed for a digital computer (see Chapter IX) so that multi-valued operators are as easy to work with as single-valued operators when the computer is used. For shorter problems where a computer is not worthwhile, rather than learning the more complicated rules, it is often helpful to sketch the curves as an aid in selecting sections to be added or multiplied.

Fig. 21 illustrates the addition of a multi-valued PWL curve to a single-valued PWL curve. For a given value of x, the values of y on the sum curve are obtained by adding each value of y_1 to the corresponding value of y_2. In order to determine the sum curve, it is necessary to carry out this addition for values of x where either curve has a breakpoint, and for one point on each end segment. To determine the correct order of the breakpoints on the sum curve, trace out the multi-valued curve starting at the left and simultaneously follow along the single-
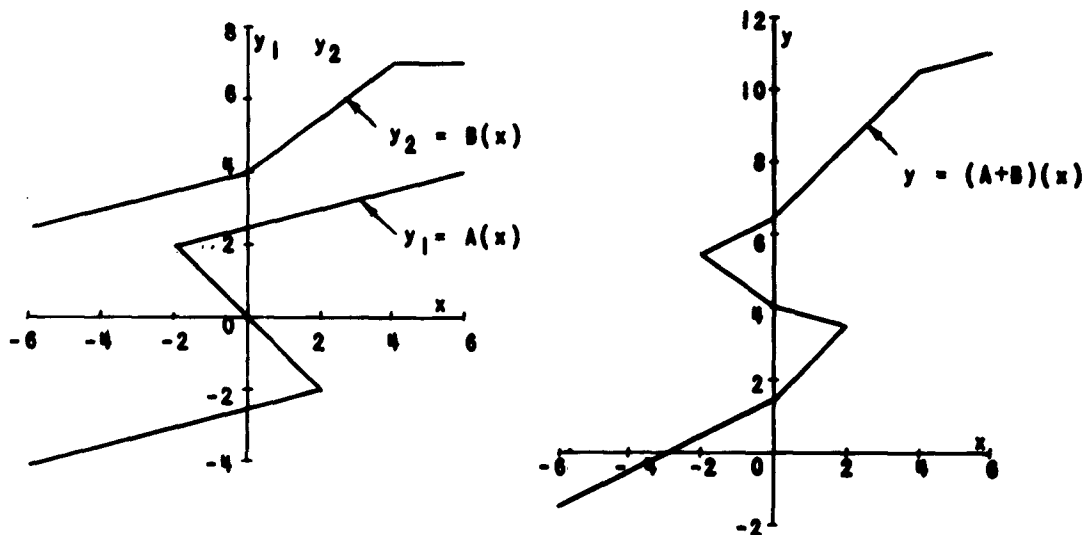
FIG. 21. ADDITION OF A MULTI-VALUED AND A SINGLE-VALUED PWL CURVE.

valued curve at the same values of x. The single-valued curve will be retraced opposite the part of the other curve that is multi-valued, so that some breakpoints may be encountered more than once.

A breakpoint at which a multi-valued curve doubles back will be referred to as a <u>corner</u> <u>point</u>. After the breakpoints of a PWL operator in slope-intercept form have been calculated, the corner points are easy to identify. The breakpoint b_k, is a corner point if

$$b_{k-1} < b_k \quad \text{and} \quad b_k > b_{k+1}$$

or if

$$b_{k-1} > b_k \quad \text{and} \quad b_k < b_{k+1}$$

When the addition of a multi-valued and a single-valued PWL curve is carried out in terms of PWL operators, the procedure is similar to that of the single-valued case except when a corner point is encountered. The addition process for single-valued operators starts with the first section of each operator and works down the column* in both operators until the last sections of both are reached. When adding a multi-valued operator A to a single-valued operator B, the direction of travel in A is always down the column, but the direction of travel in B is reversed every time a corner point of A is encountered. Moving upward in B corresponds to moving to the left on the PWL curve. Moving to the left as the addition progresses will be referred to as the <u>reverse</u> <u>mode</u> since the <u>normal</u> <u>mode</u> is moving to the right.

A more general procedure for addition of PWL operators in slope-intercept form can now be stated. Starting with the first sections of both operators,

1. Add two sections: $(q_j, r_j) + (q_k, r_k) = (q_j + q_k, \; r_j + r_k)$
2. Compare the breakpoints that follow those sections (when moving downward, b_k follows the kth section, but when moving upward, b_{k-1} follows the kth section).
3. Choose the smaller breakpoint when in the normal mode, or the larger breakpoint when in the reverse mode (if breakpoints are equal, choose both).

_____

*Moving up and down the column in the PWL operator should not be confused with moving up and down on its graph.

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4. Move to the next section of the operator in which the breakpoint was chosen.

5. If the chosen breakpoint is a corner point, reverse the direction of travel in the other operator and change the mode of addition.

6. Go back to 1. and repeat the whole process until the last sections of both operators are reached.

For the example of Fig. 21, this process gives

$$
A + B = \begin{pmatrix} -2.5, & .25 \\ 0, & -1 \\ 2.5, & .25 \end{pmatrix}\begin{matrix}2^* \\ -2^* \\ 2\end{matrix} + \begin{pmatrix} 4, & .25 \\ 4, & .75 \\ 7, & 0 \end{pmatrix}\begin{matrix}0 \\ 4\end{matrix} = \begin{pmatrix} 1.5, & .5 \\ 1.5, & 1 \\ 4, & -.25 \\ 4, & -.75 \\ 6.5, & .5 \\ 6.5, & 1 \\ 9.5, & .25 \end{pmatrix}
$$

The corner points of A have been starred. The details of the addition are given in the following table. The direction of travel is indicated by D for down and U for up, and the mode of addition is indicated by N for normal and R for reverse. The breakpoints that were chosen are marked with arrows. The table can probably be best understood by following the PWL curves in Fig. 21 while it is being read.

Sections to be Added		Direction of Travel		Mode	Breakpoints to be Compared	
A	B	A	B		A	B
1	1	D	D	N	2	0 ←—
1	2	D	D	N	2 ←—	4
2	2	D	U	R	-2	0 ←—
2	1	D	U	R	-2 ←—	--
3	1	D	D	N	--	0 ←—
3	2	D	D	N	--	4 ←—
3	3	D	D	N	--	--

If the multi-valued operator that is being added to the single-valued operator is not defined for the entire range of the independent variable, the same procedure can be applied. For Fig. 22a, we have

- 51 -

$$A + B = \begin{pmatrix} 0, & 1/2 \\ 0, & -1 \end{pmatrix} 0^* + \begin{pmatrix} 1, & 0 \\ 2, & 1/2 \\ 3, & 0 \end{pmatrix}_2^{-2} = \begin{pmatrix} 1, & 1/2 \\ 2, & 1 \\ 2, -1/2 \\ 1, & -1 \end{pmatrix}$$

When the multi-valued operator starts at the right, as in Fig. 22b, the addition process is started in the reverse mode, first adding the last section of B to the first section of A, and then moving upward in B until a corner point of A is reached. For Fig. 22b, we obtain

$$A + B = \begin{pmatrix} 0, & -1 \\ 0, & 1/2 \end{pmatrix} 0^* + \begin{pmatrix} 1, & 0 \\ 2, & 1/2 \\ 3, & 0 \end{pmatrix}_2^{-2} = \begin{pmatrix} 3, & -1 \\ 2, -1/2 \\ 2, & 1 \\ 3, & 1/2 \end{pmatrix}$$

Addition of two multi-valued curves is illustrated in Fig. 23. Selection of sections to be added is carried out by starting at the left and simultaneously tracing the two curves at the same values of i. Whenever curve A doubles back, curve B is retraced, and whenever curve B doubles back, curve A is retraced. If curves A and B represent the characteristics of PWL elements, the sum curve can be used to predict the behavior of the two elements connected in series. For example, if the initial state of the series network corresponds to point a on the

(a)   (b)
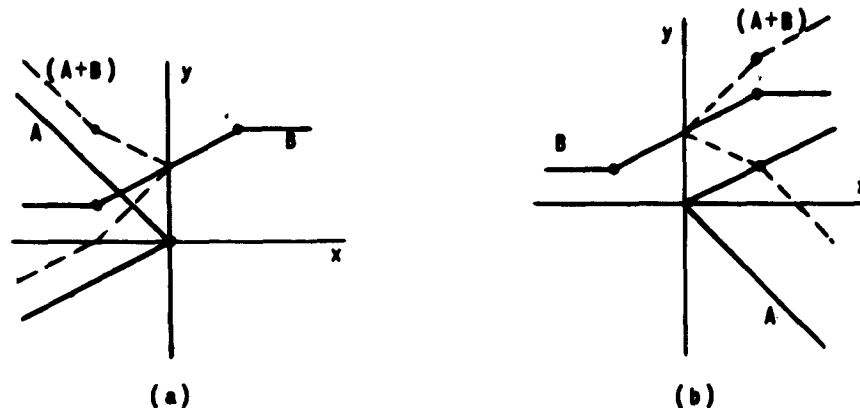
FIG. 22. ADDITION OF A CURVE DEFINED FOR A SEMI-INFINITE
RANGE OF X TO A SINGLE-VALUED CURVE.
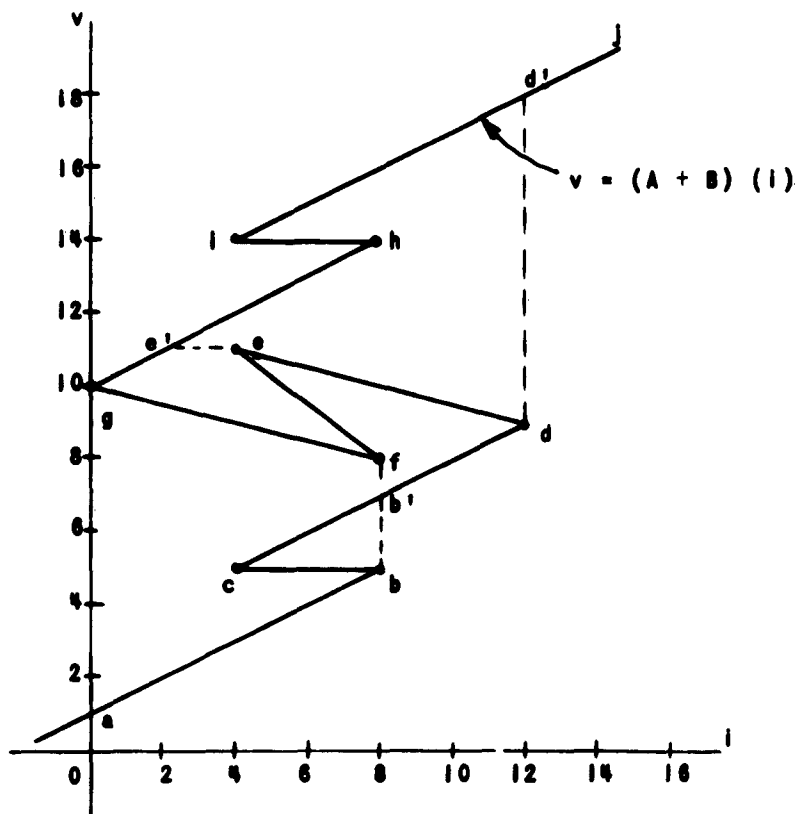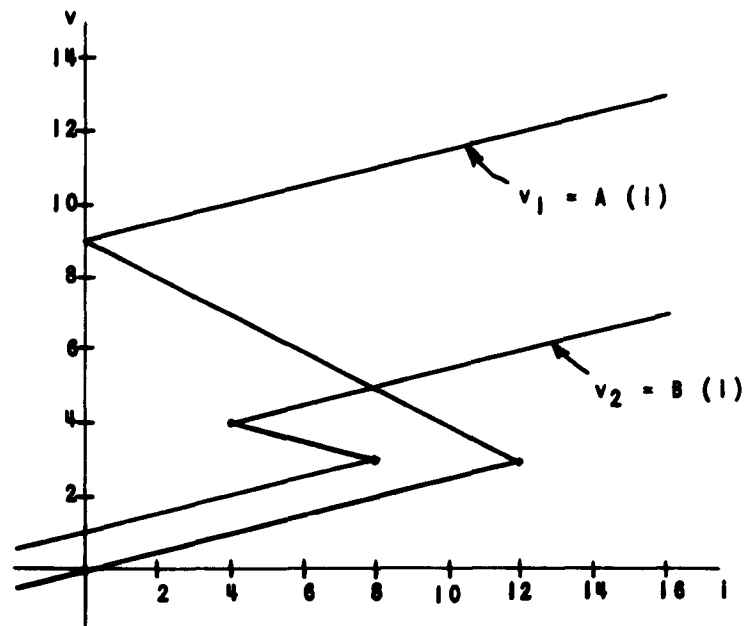
FIG. 23.   ADDITION OF MULTI-VALUED PWL CURVES.

sum curve, and we drive the network with a current source, the path a-b-b'-d-d'-j will be followed[*] as the current is continuously increased. Similarly, if we drive with a voltage source, the path a-b-c-d-e-e'-h-i-j will be followed as the voltage is continuously increased.

Addition of two multi-valued operators is similar to addition of a multi-valued and a single-valued operator except that both A and B can have corner points, so that it may be necessary to change the direction of travel in both A and B. The rules for changing directions and modes of addition are:

1. When a corner point of A is encountered, change directions in B and change modes.

2. When a corner point of B is encountered, change directions in A and change modes.

3. When corner points of both A and B are encountered simultaneously, change modes, but do not change directions.

For Fig. 23,

$$
A + B = \begin{pmatrix} 0, & 1/4 \\ 9, & -1/2 \\ 9, & 1/4 \end{pmatrix}\begin{matrix} 12^* \\ \\ 0^* \end{matrix} + \begin{pmatrix} 1, & 1/4 \\ 5, & -1/4 \\ 3, & 1/4 \end{pmatrix}\begin{matrix} 8^* \\ \\ 6^* \end{matrix} = \begin{pmatrix} 1, & 1/2 \\ 5, & 0 \\ 3, & 1/2 \\ 12, & -1/4 \\ 14, & -3/4 \\ 10, & -1/4 \\ 10, & 1/2 \\ 14, & 0 \\ 12, & 1/2 \end{pmatrix}
$$

The details of this addition are given in the following table. This example is rather complicated because every breakpoint is a corner point and a mode change is needed at every step.

---

[*] The exact nature of the dashed transitions depends on the parasitic elements that are present in the network. It will be assumed that a PWL network will remain in its present state, i.e., continue to operate on the same line segment of its characteristic, unless the terminal conditions cannot be satisfied on this line segment. In this case, the network state will change to the nearest line segment on which the new terminal conditions can be satisfied.

Sections to be Added		Direction of Travel		Mode	Breakpoints to be Compared	
A	B	A	B		A	B
1	1	D	D	N	12	8←
1	2	U	D	R	--	4←
1	3	D	D	N	12←	--
2	3	D	U	R	0	4←
2	2	U	U	N	12	8←
2	1	D	U	R	0←	--
3	1	D	D	N	--	8←
3	2	U	D	R	0	4←
3	3	D	D	N	--	--

Fig. 24 shows another example of addition of two multi-valued curves. In terms of PWL operators,

$$
A + B \;=\; \begin{pmatrix} -2, & 1/4 \\ -2, & -1/2 \\ 4, & 1/4 \end{pmatrix} \begin{matrix} 0^* \\ -8^* \end{matrix} \;+\; \begin{pmatrix} -3, & 1/4 \\ 0, & -1/2 \\ 2, & 0 \end{pmatrix} \begin{matrix} 4^* \\ -4^* \end{matrix} \;=\; \begin{pmatrix} -5, & 1/2 \\ -5, & -1/4 \\ 1, & 1/2 \\ 4, & -1/4 \\ 6, & 1/4 \end{pmatrix}
$$

In this case, the sum curve has only five sections, compared with nine in the previous example. If, instead of starting the addition process at the lower left on both curves, we start with the bottom section of A and the top section of B, we obtain the closed-loop characteristic shown with dashed lines. The solid curve correctly represents the v-i characteristic of the series connection of the PWL elements represented by A and B. The states of the network represented by the dashed curve may be thought of as transient states. If we start with the network operating at a point on the dashed curve and apply a voltage or current outside the range of the dashed curve, the operating point must shift to the solid curve. Once we are operating on the solid curve, the operating point will remain there no matter how we vary the <u>terminal</u> voltage or current. The only way to get back to an operating point on the dashed curve would be to go into the network and change the voltage across one of the series elements directly.

The addition procedure for multi-valued operators works satisfactorily if <u>one</u> of the PWL curves is undefined for a semi-infinite range of the independent variable; however, difficulties arise if both curves are of this type. If both multi-valued curves are defined for a semi-infinite
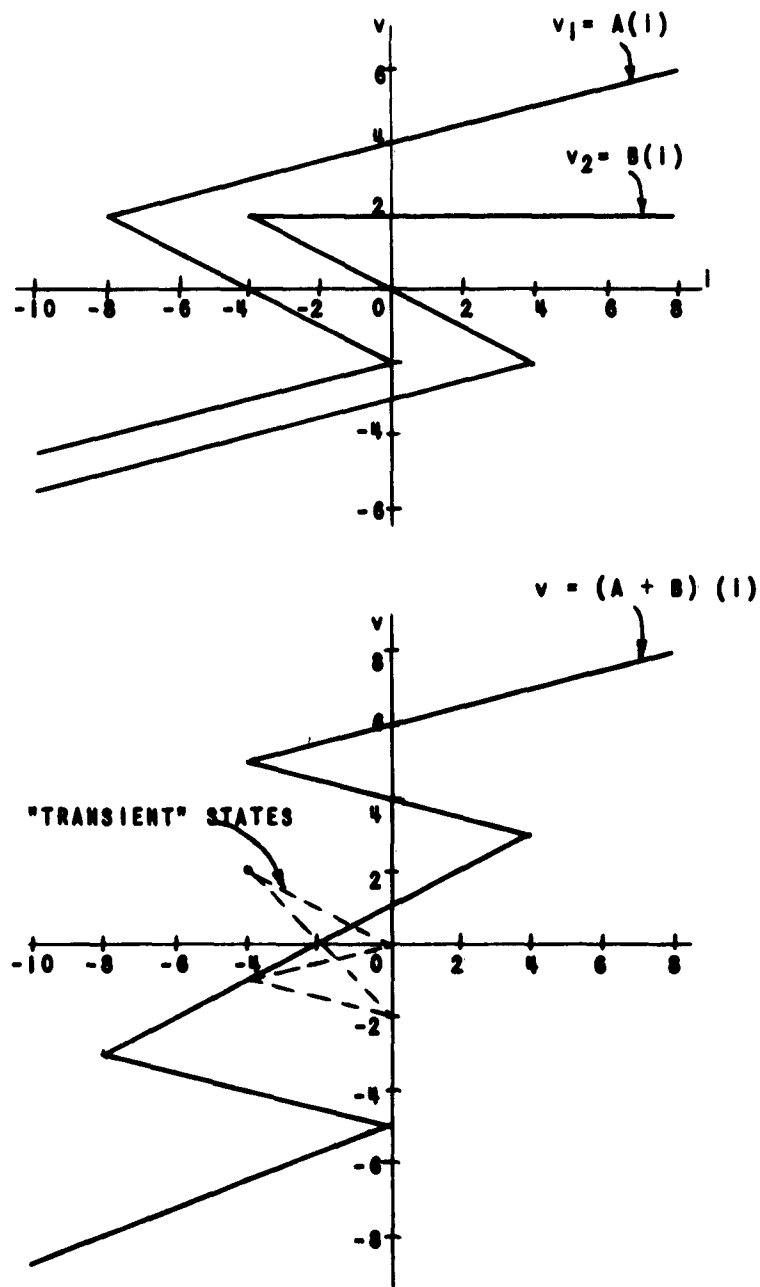
FIG. 24. ADDITION OF MULTI-VALUED PWL CURVES.

range of the independent variable, and the ranges of definition are not the same, we will say that the corresponding PWL operators are incompatible for addition. If we add the curves graphically, the sum curve may have two separate branches (Fig. 10) or may form a closed loop (Fig. 8). Even in the case where the range of definition is the same, the sum curve may have two possible values (Fig. 25). The solid curve in Fig. 25b is obtained by adding A and B in the usual manner,

$$A + B = \begin{pmatrix} 0, & 1/2 \\ 0, -3/4 \end{pmatrix} + \begin{pmatrix} 0, & 1/4 \\ 0, -1/4 \end{pmatrix} = \begin{pmatrix} 0, & 3/4 \\ 0, & -1 \end{pmatrix}$$

By listing the sections of B in the reverse order, we obtain the dashed curve

$$A + B' = \begin{pmatrix} 0, & 1/2 \\ 0, -3/4 \end{pmatrix} + \begin{pmatrix} 0, -1/4 \\ 0, & 1/4 \end{pmatrix} = \begin{pmatrix} 0, & 1/4 \\ 0, -1/2 \end{pmatrix}$$

Because of this possible ambiguity, we must be careful when carrying out addition of two PWL operators that are both undefined for some range of the independent variable. If this type of addition is required during the solution of an actual network problem, it is probably because we are working with an over-idealized model. We then have several alternate courses of action: (1) start over again using a less idealized model; (2) if the addition occurs as one of the last steps in the problem, carry it out graphically and try to interpret the result in terms of the model; or (3) carry out the addition in terms of PWL operators with the risk that the final solution may be incomplete or in error. In this case, the solution should be checked by substitution back in the original network equations. When we write an equation in terms of PWL operators, we will do so with the implicit understanding that the equation is valid only if all of the indicated algebraic operations can be carried out unambiguously.

Methods have been presented for addition of multi-valued PWL operators in slope-intercept form. Analagous methods have been developed for the breakpoint form.

J. MULTIPLICATION OF MULTI-VALUED PWL OPERATORS

The multiplication process can be generalized to handle multi-valued

(a)   (b)
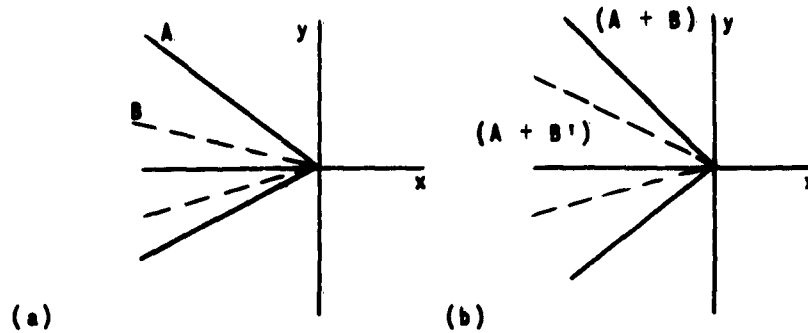
FIG. 25. ADDITION OF PWL CURVES UNDEFINED FOR x > 0.

PWL operators in the same manner that the addition process was general-
ized. The rules for selection of sections to be multiplied are com-
pletely analogous to the rules for selection of sections to be added,
except that one works with the breakpoints of A and B^{-1} instead of with
the breakpoints of A and B.

The transfer characteristic of the cascade lattice network of Fig.
26a will be derived to illustrate multiplication of a single-valued PWL
operator by a PWL operator whose inverse is multi-valued. We will
assume that $r_2 \gg r_1$ and that loading of the first lattice by the second
can be neglected. By considering the two possible states of each diode,
the transfer characteristics of the two cascaded networks are easily
seen to be

$$v_4 = \begin{pmatrix} 0,-1/2 \\ 0,\ 1/2 \end{pmatrix} (v_3) \quad \text{and} \quad v_3 = v_2 - E = \begin{pmatrix} -E,-1/2 \\ -E,\ 1/2 \end{pmatrix} (v_1)$$

$$v_4 = \begin{pmatrix} 0,-1/2 \\ 0,\ 1/2 \end{pmatrix} 0 \quad -E^* \begin{pmatrix} -E,-1/2 \\ -E,\ 1/2 \end{pmatrix} = \begin{pmatrix} -E/2,\ -1/4 \\ E/2,\ 1/4 \\ E/2,\ -1/4 \\ -E/2,\ 1/4 \end{pmatrix} (v_1)$$

This multiplication is carried out graphically in Fig. 26b.

The following example illustrates multiplication of a multi-valued
PWL operator by one whose inverse is multi-valued.

$$
\begin{pmatrix} 2, & 1 \\ -2, & -1 \\ 4, 1/2 \end{pmatrix} \begin{matrix} -2^* \\ -4^* \end{matrix} \qquad \begin{matrix} 2^* \\ 0^* \end{matrix} \begin{pmatrix} 0, & 2 \\ 4, -2 \\ -2, & 1 \end{pmatrix} = \begin{pmatrix} 2, & 2 \\ -2, & -2 \\ 4, & 1 \\ 6, & -1 \\ 3, 1/2 \end{pmatrix}
$$

Following the rules for selection of sections, the sections have been multiplied in the following order: 1st x 1st, 2nd x 1st, 3rd x 1st, 3rd x 2nd, and 3rd x 3rd.
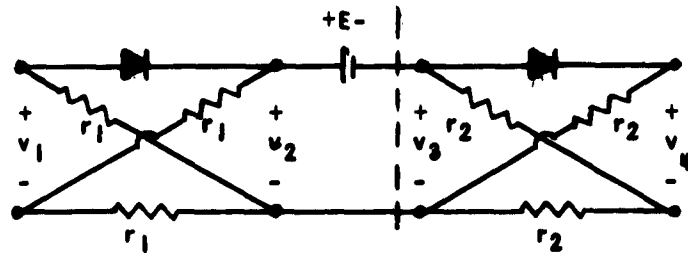
FIG. 26a. CASCADE LATTICE NETWORK.
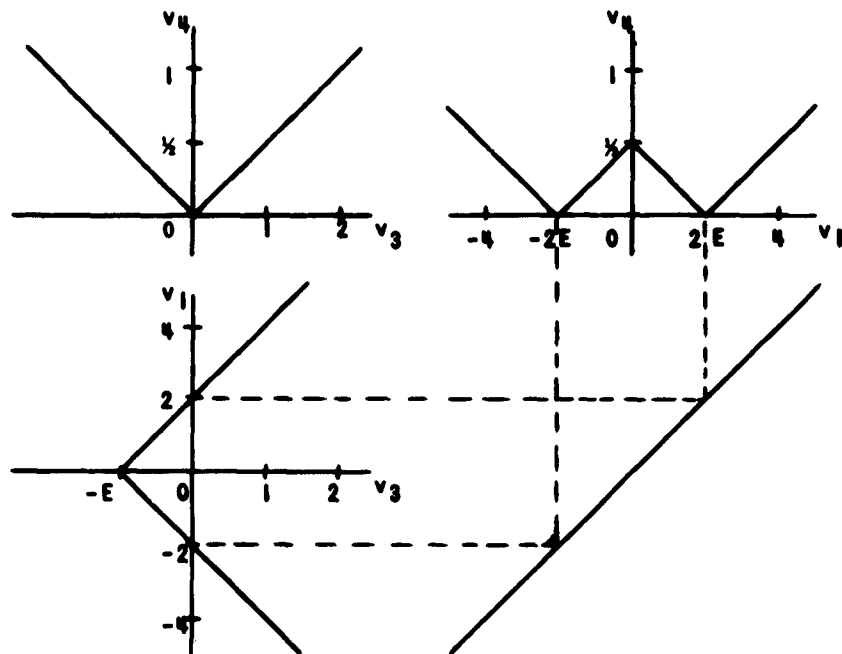
FIG. 26b. DETERMINATION OF TRANSFER CHARACTERISTIC BY GRAPHICAL MULTIPLICATION.

The product AB may be indeterminate if a section of A that has an infinite slope must be multiplied by a section of B that has a zero slope. Thus, the following product is indeterminate

$$
\begin{pmatrix} -1/2,1/2 \\ -\infty, \ \infty \\ 3/2,1/2 \end{pmatrix} \quad \begin{pmatrix} 2, \ 1 \\ 1, \ 0 \\ 0, \ 1 \end{pmatrix}
$$

This product is still indeterminate if graphical multiplication is used (Fig. 27). Since ,any point on the vertical section of A might be associated with any point on the vertical slope of B^{-1}, the product curve might take any path through the shaded indeterminate region. This difficulty can be resolved by assuming finite, non-zero slopes for A and B, and taking limits after the multiplication has been carried out.

Multiplication of PWL operators in breakpoint form has also been extended to the multi-valued case. The procedures for addition and multiplication of multi-valued PWL operators in breakpoint form have been adapted for computer use and are discussed in more detail in Section IX. C. 3.
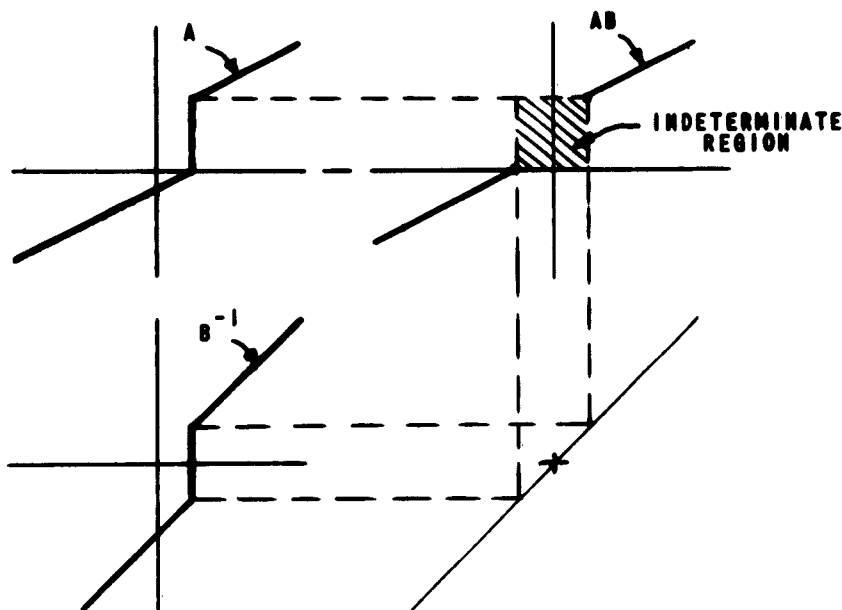
FIG. 27. INDETERMINATE CASE FOR PWL OPERATOR MULTIPLICATION.

K. SOLUTION OF PWL OPERATOR EQUATIONS

The algebra of PWL operators is similar to ordinary matrix algebra in that addition is associative and commutative, and multiplication is associative but not commutative. However, there is one important difference--the distributive law holds from both sides for matrices, but it holds only from the right for PWL operators. Some of the same techniques that are used for solving matrix equations can be used for solving PWL-operator equations, but the fact that the distributive law does not hold from the left makes it difficult or impossible to solve certain PWL-operator equations.

The inverse of the product of two PWL operators is the product of the inverses taken in reverse order just as is true for matrices. If we replace A by AB in Eq. (44), we obtain

$$(AB) \ (AB)^{-1} = I$$

Premultiplying both sides by A^{-1} and then by B^{-1} yields

$$(AB)^{-1} = B^{-1}A^{-1} \tag{61}$$

As an example of the solution of a PWL operator equation, consider the following equation which is to be solved for X:

$$A(X + B) = X + C \tag{62}$$

Subtracting $(X + B)$ from both sides yields

$$A(X + B) - (X + B) = C - B$$

We now apply the right distributive law to factor out $(X + B)$,

$$(A - I) \ (X + B) = C - B$$

Premultiplying both sides by $(A - I)^{-1}$ and subtracting B yields

$$X = (A - I)^{-1} (C - B) - B$$

Further examples of techniques for the solution of PWL-operator equations are given in Chapters IV and V.

IV. SOLUTION OF A(X + I) = BX + C

As will be seen in the next chapter, equations that can be reduced to the form

$$A(X + I) = BX + C \qquad (63)$$

occur frequently in the solution of resistive networks that contain three or more PWL resistors. Because of the importance of this equation, considerable time and effort was devoted to its solution. If the left distributive law were valid for PWL operators, it would be easy to solve Eq. (63) in terms of the basic operations that were defined in Chapter III. From the similarity of this equation and Eq. (62), it appears that solution might be possible by some clever change of variables or rearrangement of terms. Unfortunately, all attempts to solve Eq. (63) in terms of the basic operations have failed; therefore, it was necessary to devise special methods for its solution. This equation has been solved by graphical methods, by an iterative procedure, and by splitting one of the PWL operators into sections.

A. EQUIVALENT FORMS OF THE BASIC EQUATION

Eq. (63) can be written in a number of equivalent forms. Post-multiplying both sides of Eq. (63) by X^{-1}, we obtain

$$A(I + X^{-1}) = B + CX^{-1} \qquad (64)$$

If we could solve this equation for X^{-1}, we could solve the original equation for X. Substituting $Y = -(X + I)$ into Eq. (63), we obtain

$$A(-Y) = B(-Y-I) + C$$

Using Eq. (50), this becomes

$$\overline{B}(Y + I) = \overline{A}(Y) - C \qquad (65)$$

which is of the same form as the original equation. Various changes of

variable can be made in an attempt to find an equivalent equation that
is easier to solve. Unfortunately, such attempts often result in an
equation which reduces to the original form. For example, if we substi-
tute $B^{-1}YC$ for X and then postmultiply both sides of the equation by C^{-1},
we obtain

$$A(B^{-1}Y + C^{-1}) = Y + I$$

or
$$A^{-1}(Y + I) = B^{-1}Y + C^{-1} \tag{66}$$

If we let both sides of Eq. (63) operate on the variable i_1 and then
let $X(i_1) = i_2$, we obtain

$$A(i_2 + i_1) = B(i_2) + C(i_1) \tag{67}$$

Solving this equation for the relation between i_2 and i_1 is equivalent
to solving the original equation for the unknown PWL operator X.

B. SOLUTIONS FOR FIRST-ORDER PWL OPERATORS

Eq. (63) is easy to solve when any one of the three PWL operators,
A, B, or C, is of first order because it is then possible to use the
left distributive law.

If $A = (a_1, a_2)$,

$$A(X + I) = AX + a_2 = BX + C$$

and
$$X = (A - B)^{-1}(C - a_2) \tag{68}$$

If $B = (b_1, b_2)$,

$$A(X + I) = B(X + I) + C - b_2$$

and
$$X = (A - B)^{-1}(C - b_2) - I \tag{69}$$

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If $C = (c_1, c_2)$, from Eq. (64) we obtain

$$A(X^{-1} + I) = C(X^{-1} + I) + B - c_2$$

$$X^{-1} = (A - C)^{-1} (B - c_2) - I$$

and

$$X = [(A - C)^{-1} (B - c_2) - I]^{-1} \qquad (70)$$

If all three operators are first order, Eq. (63) becomes

$$(a_1, a_2) (X + I) = (b_1, b_2) X + (c_1, c_2)$$

from which

$$X = \left(\frac{c_1 + b_1 - a_1}{a_2 - b_2}, \quad \frac{c_2 - a_2}{a_2 - b_2} \right) \qquad (71)$$

C. GRAPHICAL SOLUTIONS

Equation (67) can be solved graphically to obtain the relation between i_1 and i_2. For a fixed value of i_1, we can rewrite Eq. (67) in the form

$$A(i_1 + i_2) = B[(i_1 + i_2) - i_1] + C(i_1) = B'(i_1 + i_2)$$

where B' depends on i_1. The B' curve is obtained by shifting B an amount i_1 to the right and an amount $C(i_1)$ upward. The intersection of $A(i_1 + i_2)$ and $B'(i_1 + i_2)$ determines the value of $i_1 + i_2$ that corresponds to the given value of i_1 as shown in Fig. 28. If i_1 is allowed to vary continuously, this corresponds to shifting the origin of B along the C curve. This shift can best be accomplished by plotting B on a transparent overlay, which is placed over a plot of A and C. The value of i_1 is equal to the amount that the origin of the B curve has been shifted to the right, and the corresponding value of i_2 is equal to the horizontal distance between the shifted origin and the intersection of A and B'. In order to determine completely the X curve that relates i_2 to i_1, it is necessary only to read off the values of i_2 and i_1 at the breakpoints and at one point on each end segment. X has a
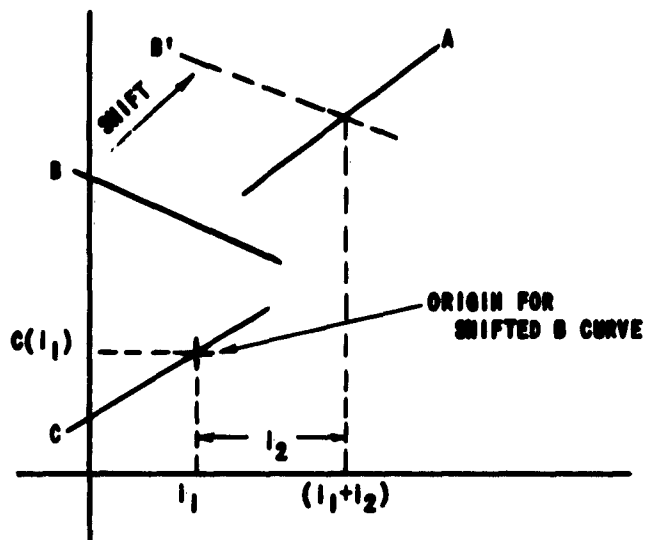
FIG. 28. DETERMINATION OF A POINT ON THE GRAPHICAL SOLUTION.

breakpoint whenever the origin of the shifted B curve lies over a break-
point of C or whenever a breakpoint of A or B' lies at the intersection
of A and B'. Fig. 29 illustrates this process for the equation

$$
\begin{bmatrix} -1, & 0 \\ 3, & 2 \\ 6, & 3 \end{bmatrix} (i_2 + i_1) = \begin{bmatrix} 0, & -3 \\ 2, & 1 \\ 4, & 3 \end{bmatrix} (i_2) + \begin{bmatrix} -3, & 7/2 \\ -1, & 3 \\ 2, & 0 \\ 4, & 0 \end{bmatrix} (i_1)
$$

or equivalently,

$$
\begin{pmatrix} 1/2, & 1/2 \\ 1, & 1/3 \end{pmatrix} (X + I) = \begin{pmatrix} -3, & 2 \\ -1, & 1 \end{pmatrix} X + \begin{pmatrix} 11/4, -1/4 \\ 2, & -1 \\ 0, & 0 \end{pmatrix} \tag{72}
$$

To get a "feel" for the process, it is suggested that the reader plot B
on a transparent overlay, slide it along C, and note where it intersects
A. The positions of B that determine the breakpoints of X are indicated
with dashed lines, and the corresponding origins of the B curve are
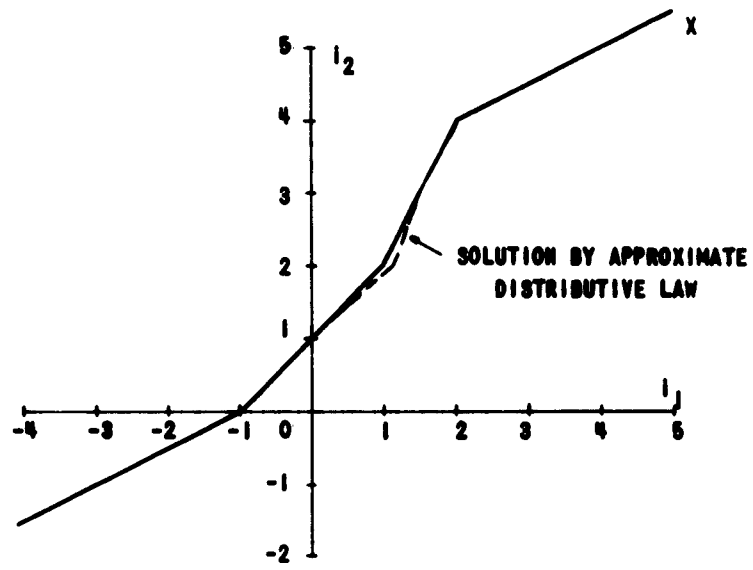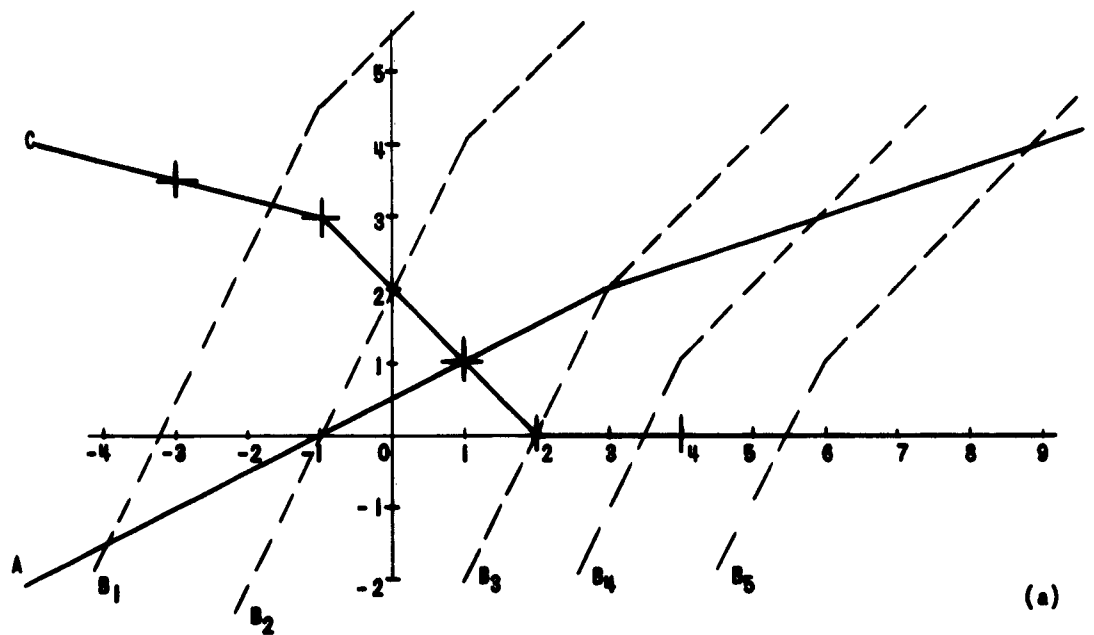
$$X = \begin{bmatrix} -3, & -1 \\ 1, & 0 \\ 1, & 2 \\ 2, & 4 \\ 4, & 5 \end{bmatrix} = \begin{pmatrix} 1/2, & 1/2 \\ 1, & 1 \\ 0, & 2 \\ 3, & 1/2 \end{pmatrix}$$

FIG. 29. GRAPHICAL SOLUTION OF EQ. (72)

indicated with heavy crosses. To form X in breakpoint form, the values
of i_1 and i_2 that are read off the graph at the breakpoints are listed
in order, and then X can be converted to slope-intercept form if desired.
To verify that the solution is correct, we substitute X back into Eq. (72):

$$\begin{pmatrix} 1/2, & 1/2 \\ 1, & 1/3 \end{pmatrix} \begin{matrix} -1 \\ 3 \\ 3 \\ 6 \end{matrix} \begin{pmatrix} 1/2, & 3/2 \\ 1, & 2 \\ 0, & 3 \\ 3, & 3/2 \end{pmatrix} = \begin{pmatrix} -3, & 2 \\ -1, & 1 \end{pmatrix} \begin{matrix} 0 \\ 2 \\ 2 \\ 4 \end{matrix} \begin{pmatrix} 1/2, & 1/2 \\ 1, & 1 \\ 0, & 2 \\ 3, & 1/2 \end{pmatrix} + \begin{pmatrix} 11/4, -1/4 \\ 2, & -1 \\ 0, & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3/4, & 3/4 \\ 1, & 1 \\ \hline 1, & 1 \\ 2, & 1/2 \end{pmatrix} = \begin{pmatrix} -2, & 1 \\ -1, & 2 \\ \hline 1, & 2 \\ 2, 1/2 \end{pmatrix} \begin{matrix} -1 \\ \\ 2 \end{matrix} + \begin{pmatrix} 11/4, -1/4 \\ 2, & -1 \\ 0, & 0 \end{pmatrix} \begin{matrix} -1 \\ \\ 2 \end{matrix}$$

$$= \begin{pmatrix} 3/4, & 3/4 \\ 1, & 1 \\ 2, & 1/2 \end{pmatrix}$$

Fig. 30 illustrates the graphical solution of

$$\begin{pmatrix} -2, & 1/2 \\ -3, & 1 \end{pmatrix} (i_2 + i_1) = \begin{pmatrix} +1, & -1 \\ 2, & -2 \end{pmatrix} (i_2) + \begin{pmatrix} -1, & 1 \\ 1/2, & 1/2 \end{pmatrix} (i_1) \quad (73)$$

In this example, the breakpoint of B intersects A <u>twice</u> as B is slid
along C.

For a given value of i_1, the shifted B curve may not intersect the
A curve at all or it may intersect it several times. Thus, for certain
values of i_1, the solution curve may be undefined or it may be multi-
valued. For the network of Fig. 11a,

$$v_1 = \left[(1, -1) + \begin{pmatrix} 0, 1 \\ 0, 0 \end{pmatrix} 0 + \begin{pmatrix} 0, 0 \\ -7, 2 \end{pmatrix} 7/2 \right] (i_1 - 2v_1) + \begin{pmatrix} 0, 1 \\ 0, 0 \end{pmatrix} (i_1)$$

$$(74)$$

(a)

(b)
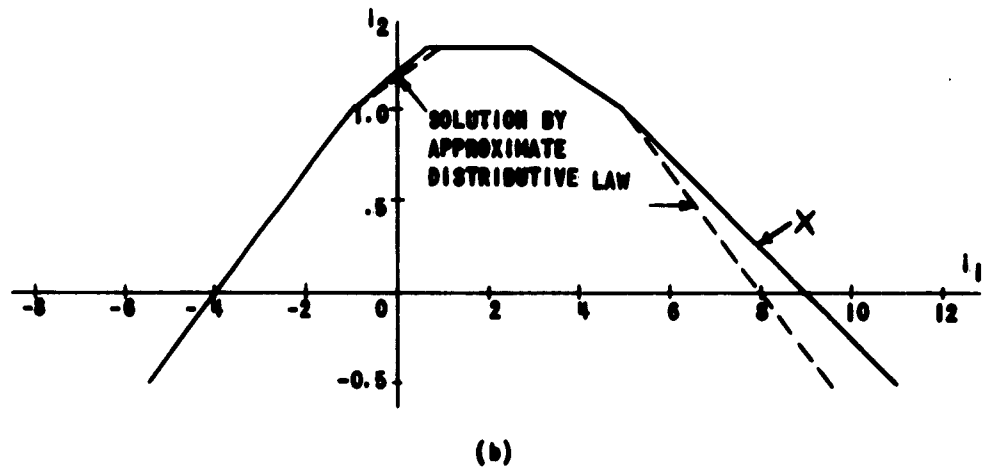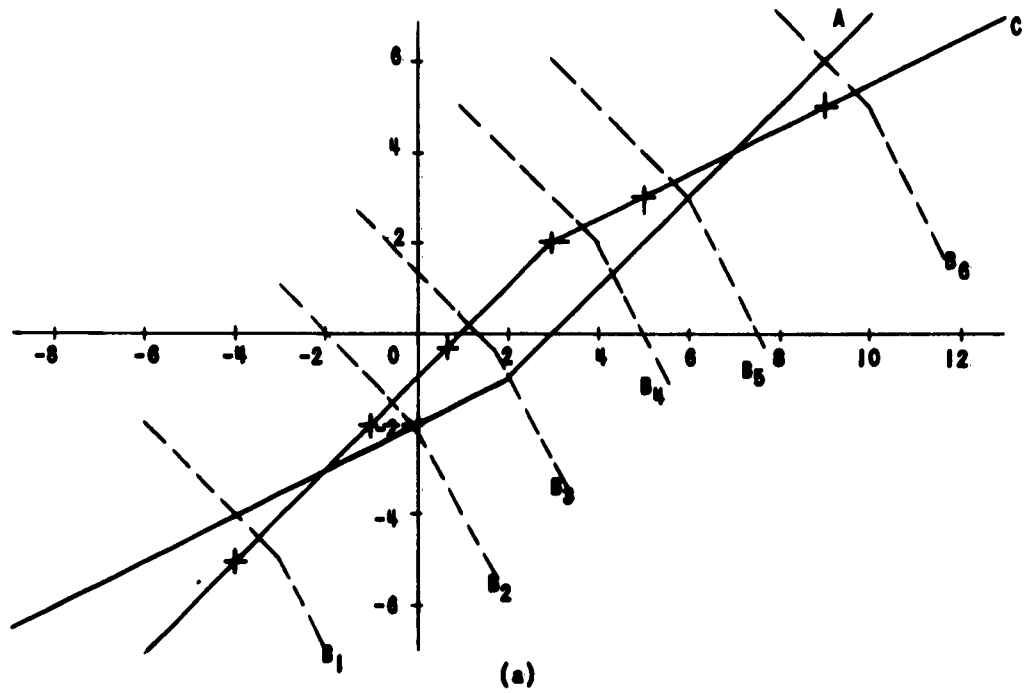
$$X = \begin{bmatrix} -4, & 0 \\ -1, & 1 \\ 2/3, & 4/3 \\ 3, & 4/3 \\ 5, & 1 \\ 8, & 0 \end{bmatrix} = \begin{pmatrix} 4/3, & 1/3 \\ 6/5, & 1/5 \\ 4/3, & 0 \\ 11/6, & -1/6 \\ 8/4, & -1/4 \end{pmatrix}$$

FIG. 30.   GRAPHICAL SOLUTION OF EQ. (73).

If we let $v_1 = -\frac{1}{2} X (i_1)$, Eq. (74) can be reduced to

$$\begin{pmatrix} -1, & 0 \\ -1, & 1 \\ 6, & -1 \end{pmatrix} (X + I) = \frac{1}{2} X + \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} \qquad (75)$$

Although this equation could be solved by Eq. (69), we will use the graphical procedure instead (Fig. 31a). As the origin of the B curve slides along C, B first moves up through positions B_1 and B_2 until B_3 is reached, and then moves down again through positions B_4 and B_5. Until B_2 is reached, there is only one intersection with A, but between B_2 and B_3, there are three intersections, and beyond B_4 there is only one intersection again. Thus, $X(i_1)$ is triple-valued for $-2 < i_1 < 2$ and single-valued elsewhere. The solution has two distinct branches (the solid curves in Fig. 31b). If we multiply this solution by $-1/2$, we obtain the curve of v_1 vs. i_1 shown in Fig. 11c.

If better accuracy is needed than can be obtained with the graphical procedure, Eq. (71) can be used to calculate the numerical values for each section of X after the appropriate sections of A, B, and C have been selected graphically. For example, to calculate the first section of X in Fig. 30, we take

$$(a_1, a_2) = (-2, 1/2) \qquad (b_1, b_2) = (1, -1) \qquad (c_1, c_2) = (-1, 1)$$

and apply Eq. (71) to obtain $X_1 = (4/3, 1/3)$, which checks the graphical solution.

Since the basic operation involved in the graphical procedure is the determination of points of intersection of straight lines, in principle it should be possible to describe the procedure in such a manner that it can be carried out numerically. In the case where A, B, C, and X are all monotonic (as in Fig. 29), the numerical procedure is relatively straightforward. After Eq. (71) has been used to calculate a section of X from appropriate sections of A, B, and C, a simple comparison scheme is used to determine whether a breakpoint of A, B, or C will be encountered next. When some of the curves are multi-valued, the decision

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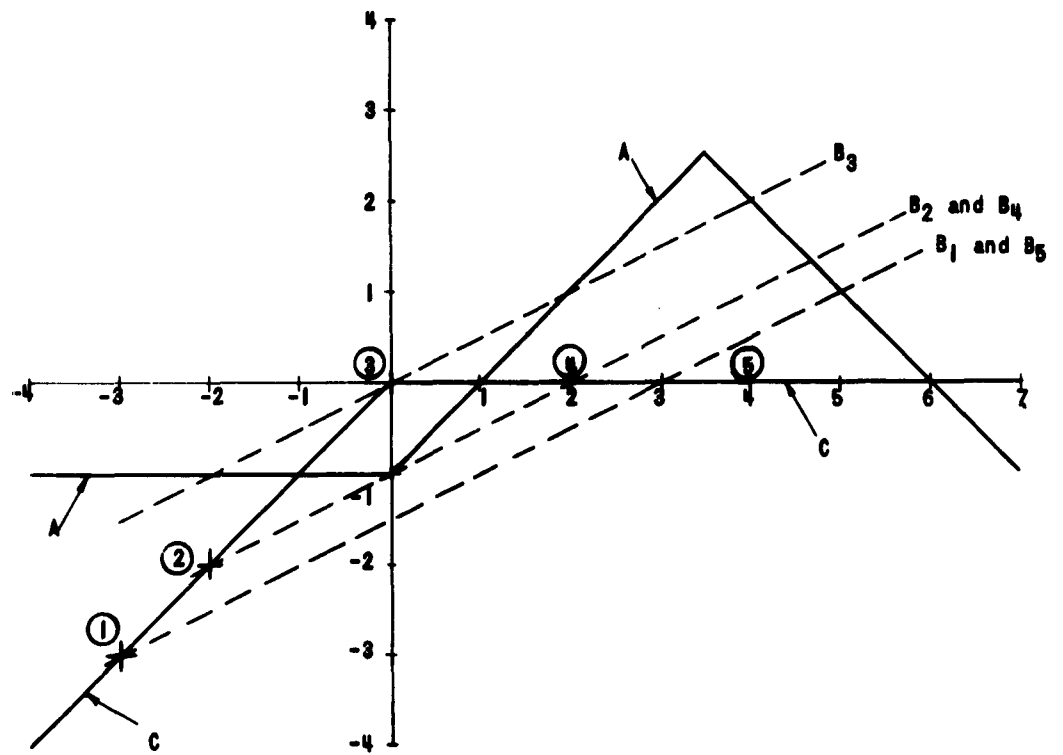
(a)
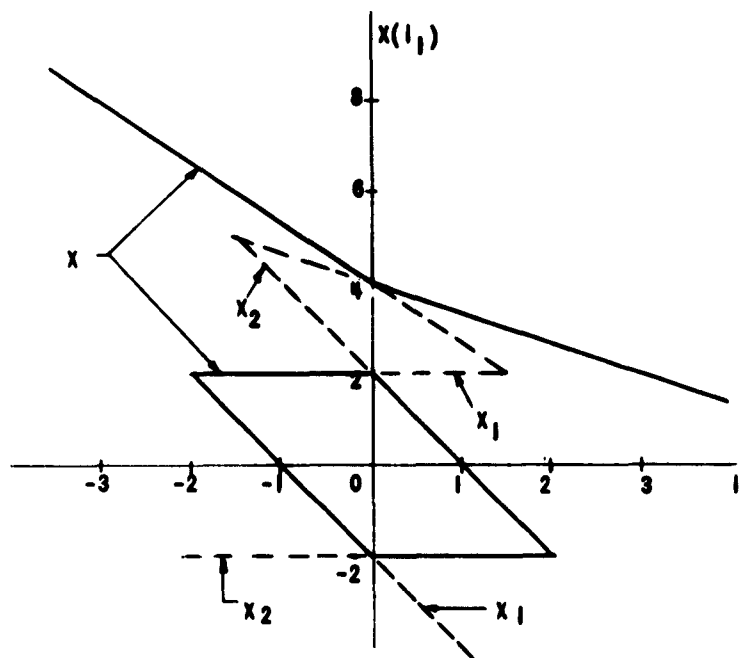
(b)

FIG. 31. GRAPHICAL SOLUTION OF EQ. (75)

rule for determining the successive sections of A, B, and C that are to be used is so complicated that it is not practical to formulate it in detail. For this reason, the graphical procedure would be difficult to adapt for use with a digital computer.

D. APPROXIMATE SOLUTION

The basic reason that Eq. (63) is difficult to solve is that the left distributive law is invalid. If we had an approximation to the left distributive law, we could obtain an approximate solution of the equation. We would like to obtain an approximate distributive law of the form

$$A(X + Y) \approx A'(X) + A'(Y) \tag{76}$$

When $X = Y$, Eq. (76) becomes

$$A(2X) \approx 2A'(X)$$

If we choose A' so that Eq. (76) is exact when $X = Y$, we obtain

$$A' = (\tfrac{1}{2} \cdot A \cdot 2) = (0, \tfrac{1}{2}) A(0, 2) \tag{76a}$$

Using Eq. (76) to find an approximate solution to Eq. (63), we obtain

$$A(X + I) = BX + C \approx A'(X) + A' \tag{77}$$

and

$$X \approx (A' - B)^{-1} (C - A')$$

This solution is exact if A is linear. For Eq. (72),

$$A' = \tfrac{1}{2} \cdot \begin{pmatrix} 1/2, 1/2 \\ 1, 1/3 \end{pmatrix} \cdot 2 = \begin{pmatrix} 1/4, 1/2 \\ 1/2, 1/3 \end{pmatrix} \quad X \approx \begin{bmatrix} -3, & -1 \\ -1, & 0 \\ 1/2, & 3/2 \\ 19/18, & 2 \\ 3/2, & 3 \\ 2, & 4 \\ 4, & 5 \end{bmatrix} \tag{78}$$

And for Eq. (73),

$$A' = \frac{1}{2} \cdot \begin{pmatrix} -2, & \frac{1}{2} \\ -3, & 1 \end{pmatrix} \cdot 2 = \begin{pmatrix} -1, & \frac{1}{2} \\ -3/2, & 1 \end{pmatrix} \qquad X \approx \begin{pmatrix} 4/3, & 1/3 \\ 7/6, & 1/6 \\ 4/3, & 0 \\ 11/6, & -1/6 \\ 8/3, & -1/3 \end{pmatrix} \qquad (79)$$

In these two examples, the results obtained by the approximate distributive law are very close to the correct solution as shown by the dashed lines in Figs. 29b and 30b.

The approximate left distributive law can be applied to the approximate solution of more complicated PWL operator equations. The results will not always be as good as in the above examples, but solution curves of the correct shape will ger.rally be obtained.

E. ITERATIVE SOLUTION

After an approximate solution to Eq. (63) has been obtained by using the approximate distributive law, this approximation can be improved by using an iterative procedure. If we solve the left side of Eq. (63) for X, we obtain

$$X = B^{-1}[A(X + I) - C] \qquad (80)$$

which suggests using the iteration

$$X_{k+1} = B^{-1}[A(X_k + I) - C] \qquad (81)$$

where X_k is the kth approximation to X.

A sufficient condition for convergence of this iteration will now be derived. Subtracting Eq. (80) from Eq. (81) yields

$$X_{k+1} - X = B^{-1}[A(X_k + I) - C] - B^{-1}[A(X + I) - C]$$

If A and B are first order, the distributive law of Eq. (57) can be applied, and

$$X_{k+1} - X = B^{-1}AX_k - B^{-1}AX$$

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If

$$A = (a_1, a_2) \text{ and } B = (b_1, b_2)$$

$$B^{-1}A = \left(-\frac{B_1}{b_2}, \frac{1}{b_2}\right) \left(a_1, a_2\right) = \left(\frac{a_1-b_1}{b_2}, \frac{a_2}{b_2}\right)$$

and

$$(X_{k+1} - X_k) = (0, a_2/b_2) (X_k - X) \tag{82}$$

Starting with an initial approximation X_0 applying Eq. (82) n times yields

$$(X_n - X) = \left(0, \frac{a_2}{b_2}\right)^n (X_0 - X) = \left(\frac{a_2}{b_2}\right)^n (X_0 - X) \tag{83}$$

In words, the error in the nth approximation is $(a_2/b_2)^n$ times the error in the initial approximation. If $|a_2/b_2| < 1$, $(a_2/b_2)^n \to 0$ as $n \to \infty$, and $(X_n - X) \to 0$. If X_n and X are defined for the same range of independent variable, then $X_n \to X$. Thus for any reasonable[*] initial approximation to X, the iteration will converge if $|a_2| < |b_2|$. The ratio $|a_2/b_2|$ gives some measure of how rapidly the iteration will converge. In general, the smaller the ratio $|a_2/b_2|$ and the closer the initial approximation, the more rapid will be the convergence.

We will now consider convergence when A and B are of higher order. If every section of X converges to the correct value, then X must converge to the correct value. If we knew which sections of A and B corresponded to each section of X, we could apply the above procedure to test the convergence of each section. However, we usually do not know which sections of A and B correspond to a given section of X, so we will consider the possibility that any section of A might be paired with any section of B during the calculation. Since convergence of a given section of X depends on the ratio of the slope of a section of A to the slope of a section of B, the worst possible situation occurs when the slope of A is maximum and the slope of B is minimum. Let $|r_A|_{max}$ be the maximum absolute value of all the slopes of A and let $|r_B|_{min}$ be the

---

[*] It would not be reasonable, for example, to approximate a single-valued PWL operator, which is defined for all values of i_1, by a multi-valued operator defined only for $i_1 > 0$. In this case, the iteration would still converge, but not necessarily to the correct value.

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minimum absolute value of all the slopes of B. Then if $|r_A|_{max} < |r_B|_{min}$, convergence of all sections of X is assured, provided that the initial approximation is sufficiently close. The iteration may converge in some cases even if $|r_A|_{max} \geq |r_B|_{min}$ since the worst combination of slopes will not necessarily be encountered during the solution of a problem.

In some cases where the iteration of Eq. (81) fails to converge, the following iteration may be used instead:

$$X_{k+1} = A^{-1} (BX_k + C) - I \qquad (84)$$

This iteration converges if $|r_B|_{max} < |r_A|_{min}$. If the iterations of Eqs. (81) and (84) both fail to converge, a similar iteration may converge for one of the equivalent forms of Eq. (63). For example, if we solve Eq. (64) for X^{-1} by an iteration similar to Eq. (81), the roles of B and C are interchanged so the new condition for convergence is $|r_A|_{max} < |r_C|_{min}$.

Although these iterative methods are cumbersome to carry out by hand, they are well suited for use with a digital computer. A number of examples of Eq. (63) have been worked on the Burroughs 220 using the iteration of Eq. (81). The details of the computer programs are discussed in Chapter IX. Several different initial approximations were tried for each equation. The initial approximation obtained by the approximate distributive law, Eq. (77), generally converges most rapidly, but in many examples, a much cruder approximation such as $X_o = 0$ converges almost as fast. Since the initial and final segments of X are not difficult to calculate, another useful initial approximation is a second-order PWL operator that agrees with the initial and final segments of X.

The equation

$$\begin{pmatrix} 0, & .1 \\ 0, & 1 \end{pmatrix} (X + I) = \begin{pmatrix} 3, & -10 \\ 3, & -2 \end{pmatrix} X + \begin{pmatrix} 0, & 1 \\ 0, & .1 \end{pmatrix} \qquad (85)$$

has $|r_A|_{max} / |r_B|_{min} = 1/2$; therefore, rapid convergence of the iterative solution may be expected. Starting with the initial approximation, $X_o = 0$, successive approximations X_1, X_2, and X_3 are plotted in Fig. 32. Note that the maximum error is cut in half with each iteration. The final solution,
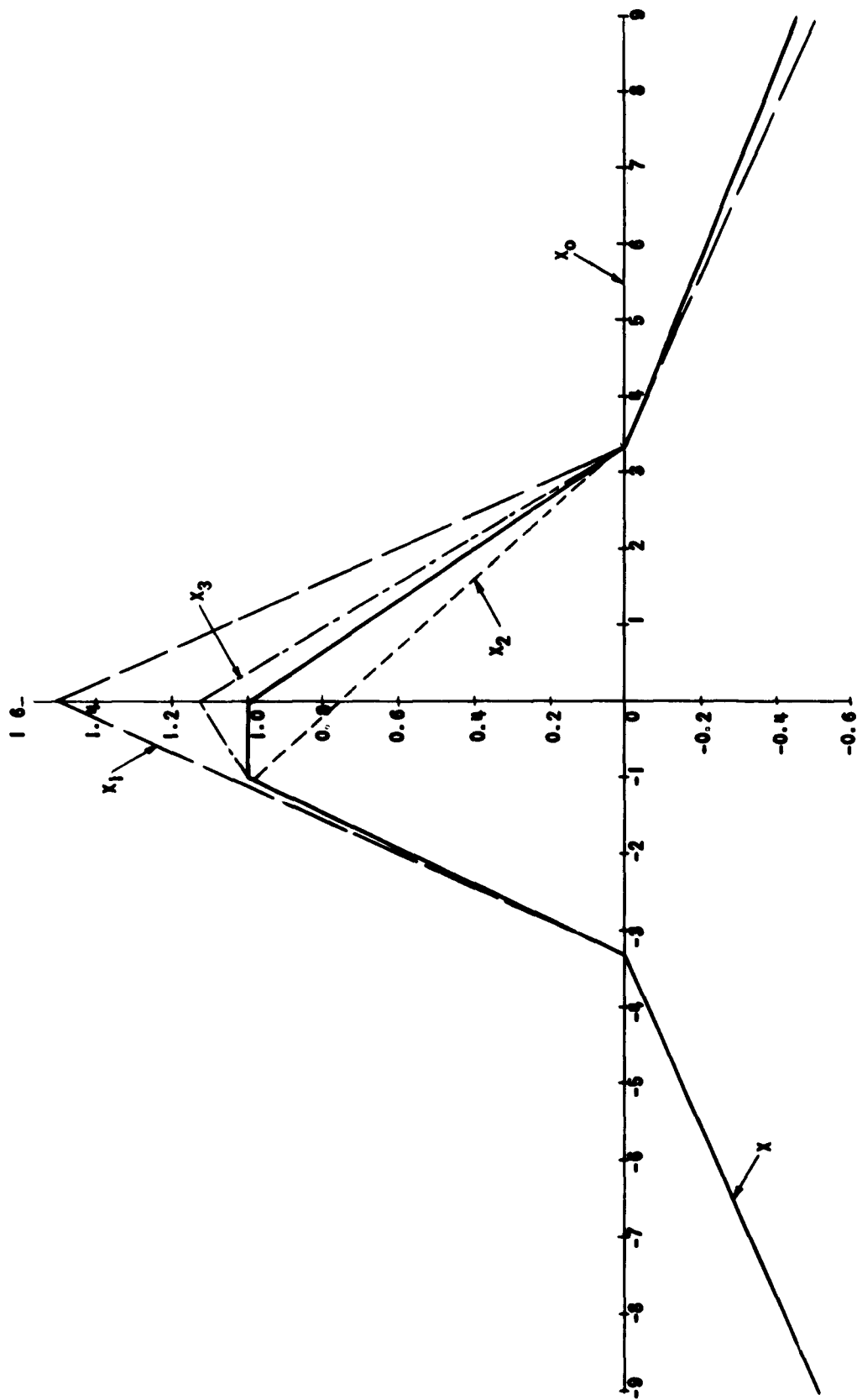
FIG. 32. ITERATIVE SOLUTION OF EQ. (85).

$$X = \begin{bmatrix} -6.7, & -0.3 \\ -10/3, & 0 \\ -1, & 1 \\ 0, & 1 \\ 10/3, & 0 \\ 7, & -0.3 \end{bmatrix} = \begin{pmatrix} 30/101, & 9/101 \\ 10/7, & 3/7 \\ 1, & 0 \\ 1, & -0.3 \\ 3/11, & -9/110 \end{pmatrix}$$

is obtained correct to four decimal places in the breakpoints on the 13th iteration. Similar accuracy is obtained in 10 iterations when the initial approximation is calculated by the approximate distributive law.

For Eq. (72), $|r_A|_{max}/|r_B|_{min} = 1/2$. Starting with the initial approximation of Eq. (78), the solution converges to four-place accuracy in seven iterations, and starting with $X_0 = 0$, nine iterations are required. For Eq. (73), the iteration converges for all sections of X except the last one. For this section, $r_A/r_B = -1$. Starting with the approximation of Eq. (79), after 15 iterations the solution oscillates between the values

$$\begin{bmatrix} -4, & 0 \\ -1, & 1 \\ 2/3, & 4/3 \\ 3, & 4/3 \\ 5, & 1 \\ 11, & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -4, & 0 \\ -1, & 1 \\ 2/3, & 4/3 \\ 3, & 4/3 \\ 11, & 0 \end{bmatrix}$$

A similar oscillation occurs after 21 iterations if $X_0 = 0$ is used. In cases like this, convergence can be obtained by applying an averaging procedure after each iteration.

F. THE SECTION METHOD

As shown in Section IV. B, Eq. (63) has a simple solution if A, B, or C is of first order. When all the operators are of higher order, it is possible to break one of the operators into sections, solve the equation for each section, determine the region of validity for each partial solution, and then patch the partial solutions together to form the complete solution.

If C is of second order and single-valued, we can shift the break-point of C to the origin by a change of variables and then write Eq. (67) in the form

$$A(i_1 + i_2) = B(i_2) + \begin{pmatrix} 0, & c_1 \\ 0, & c_2 \end{pmatrix} (i_1) \tag{86}$$

If $i_1 \leq 0$, this becomes

$$A(i_1 + i_2) = B(i_2) + (0, c_1)(i_1)$$

and, by Eq. (70)

$$i_2 = [(A - c_1)^{-1}(B - c_1) - I]^{-1}(i_1) = X_1(i_1) \tag{87}$$

Similarly, for $i_1 \geq 0$,

$$i_2 = [(A - c_2)^{-1}(B - c_2) - I]^{-1}(i_1) = X_2(i_1) \tag{88}$$

If $X_1(0)$ and $X_2(0)$ are single-valued

$$X_1 \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} (i_1) = \begin{cases} X_1(i_1) & \text{if } i_1 \leq 0 \\ X_1(0) & \text{if } i_1 \geq 0 \end{cases} \tag{89}$$

$$X_2 \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} (i_1) = \begin{cases} X_2(0) & \text{if } i_1 \leq 0 \\ X_2(i_1) & \text{if } i_1 \geq 0 \end{cases} \tag{90}$$

Since $i_2(0) = X_1(0) = X_2(0)$, we can express i_2 in the form

$$i_2 = X(i_1) = \left[X_1 \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} + X_2 \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \right] (i_1) - X_1(0) \tag{91}$$

For Eq. (85),

$$X_1 = \begin{pmatrix} 30/101, 9/101 \\ 10/7, & 3/7 \\ 1, & 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 10/7, & 0 \\ 1, - 3/10 \\ 3/11, -9/110 \end{pmatrix}$$

$$X = X_1 \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} + X_2 \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} - X_1(0)$$

$$= \begin{pmatrix} 30/101, & 9/101 \\ 10/7, & 3/7 \\ 1, & 0 \end{pmatrix} \begin{matrix} -10/3 \\ -1 \end{matrix} + \begin{pmatrix} 1, & 0 \\ 1, & -3/10 \\ 3/11, & -9/110 \end{pmatrix} \begin{matrix} 0 \\ 10/3 \end{matrix} - (1,0) = \begin{pmatrix} 30/101, & 9/101 \\ 10/7, & 3/7 \\ 1, & 0 \\ 1, & -3/10 \\ 3/11, & -9/110 \end{pmatrix}$$

which checks the result obtained by the iterative method.

We will now extend the above procedure in order to solve Eq. (63) when C is of nth order and single-valued. We can separate Eq. (67) into a series of n equations, each valid over a specified range of i_1:

$$A (i_1 + i_2) = B (i_2) + C_k (i_1) \qquad b_{k-1} \leq i_1 \leq b_k \qquad (k = 1,2, \ldots, n) \quad (92)$$

where C_k is the kth section of C and b_k is the kth breakpoint of C. The solutions to these equations are

$$i_2 = X_k (i_1) = [(A - C_k)^{-1} (B - c_k) - I]^{-1} (i_1) \qquad (k = 1,2, \ldots, n) \quad (93)$$

where c_k is the slope of C_k. We now define

$$Q_k = \begin{pmatrix} b_{k-1}, & 0 \\ 0, & 1 \\ b_k, & 0 \end{pmatrix} \qquad (94)$$

as shown in Fig. 33. If X_k is single-valued at b_{k-1} and b_k,

$$X_k Q_k (i_1) = \begin{cases} X_k(b_{k-1}) & i_1 \leq b_{k-1} \\ X_k(i_1) & b_{k-1} \leq i_1 \leq b_k \\ X_k(b_k) & b_k \leq i_1 \end{cases} \qquad (95)$$

Since $i_2(b_k) = X_k(b_k) = X_{k+1}(b_k)$, i_2 can be expressed in the form

$$i_2 = X(i_1) = \sum_{k=1}^{n} X_k Q_k (i_1) - \sum_{k=1}^{n-1} X_k (b_k) \qquad (96)$$

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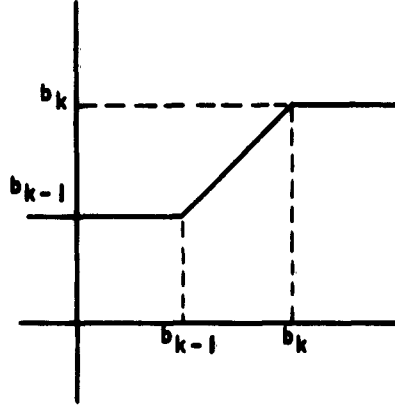
FIG. 33. PLOT OF Q_k.

We will now show that this equation gives the correct value of i_2 for every range of i_1.

If $b_{j-1} \le i_1 \le b_j$, by using Eq. (95) we obtain

$$i_2 = \sum_{k=1}^{j-1} X_k \, (b_k) + X_j \, (i_1) + \sum_{k=j+1}^{n} X_k \, (b_{k-1}) - \sum_{k=1}^{n-1} X_k \, (b_k) \qquad (96a)$$

Since
$$\sum_{k=j+1}^{n} X_k \, (b_{k-1}) = \sum_{k=j}^{n-1} X_{k-1} \, (b_k) = \sum_{k=j}^{n-1} X_k \, (b_k)$$

Eq. (96a) reduces to $i_2 = X_j \, (i_1)$, which is the correct solution for the specified range of i_1.

For Eq. (72),

$$X = X_1 \, Q_1 + X_2 \, Q_2 + X_3 \, Q_3 - X_1 \, (b_1) - X_2 \, (b_2)$$

$$= \begin{pmatrix} 1/2, \ 1/2 \\ 3/4, 7/20 \\ -9/8, \ 7/8 \end{pmatrix} \begin{pmatrix} 0, \ 1 \\ -1, \ 0 \end{pmatrix} + \begin{pmatrix} 1, \ 1 \\ 0, \ 2 \end{pmatrix} \begin{pmatrix} -1, \ 0 \\ 0, \ 1 \\ 2, \ 0 \end{pmatrix} + \begin{pmatrix} 7/3, \ 1/3 \\ 3, \ 1 \\ 3, \ 1/2 \end{pmatrix} \begin{pmatrix} 2,0 \\ 0,1 \end{pmatrix} - (0,0)-(4,0)$$

$$= \begin{pmatrix} 1/2,1/2 \\ 0,\ 0 \end{pmatrix}_{-1} + \begin{pmatrix} 0,\ 0 \\ 1,\ 1 \\ 0,\ 2 \\ 4,\ 0 \end{pmatrix}_{\substack{-1 \\ 1 \\ 2}} + \begin{pmatrix} 4,\ 0 \\ 3,1/2 \end{pmatrix}_2 - (4,0) = \begin{pmatrix} 1/2,1/2 \\ 1,\ 1 \\ 0,\ 2 \\ 3,1/2 \end{pmatrix}$$

which checks the graphical solution of Fig. 29.

The section method can be applied directly to any equation of the form of Eq. (63) provided that C is single-valued and all the partial solutions are single-valued at the breakpoints of C. If C is multi-valued, it may be possible to apply the section method to one of the equivalent forms of Eq. (63), e.g., if B is single-valued Eq. (64) could be used. If some of the partial solutions are multi-valued at the breakpoints, the section method can still be used, but the partial solutions must be combined graphically because the indicated multiplications in Eq. (96) cannot be carried out. As an example, we will solve Eq. (75) by the section method. The partial solutions

$$X_1 = \begin{pmatrix} 4,-4/3 \\ 2,\ 0 \\ -2,\ -2 \end{pmatrix} \qquad (i_1 \leq 0)$$

and

$$X_2 = \begin{pmatrix} -2,\ 0 \\ 2,\ -2 \\ 4,-2/3 \end{pmatrix} \qquad (i_1 \geq 0)$$

are plotted in Fig. 31 with dashed lines. The complete solution is formed by taking X_1 for $i_1 \leq 0$, and X_2 for $i_1 \geq 0$.

The section method has also been programmed for a digital computer (see Section IX. C. 4). It is superior to the iterative method because it is much faster and there are no problems with convergence.

G. TRIVOLUTION

Although Eq. (63) generally cannot be solved in terms of the basic operations, it can always be solved by the graphical method, and in most cases the equation or one of its equivalent forms can also be solved by the iterative method or by the section method. Since the operation of solving this equation occurs frequently in the solution of networks which contain three or more PWL resistors, it is convenient to have a name and symbol for the operation. We will write $X = A*B*C$ as a symbol for the solution of $A(X + I) = BX + C$, and since there are three operands involved, we will call the operation trivolution.

Trivolution has several useful algebraic properties. In terms of trivolution, the solution to Eq. (64) is $X^{-1} = A*C*B$. Since $X = A*B*C$, we have the relationship

$$(A*B*C)^{-1} = A*C*B \qquad (97)$$

From Eq. (65),

$$Y = -(X + I) = \overline{B} * \overline{A} * -C$$

and substitution of $A*B*C$ for X yields

$$A*B*C + I = -\overline{B} * \overline{A} * -C \qquad (98)$$

From Eq. (66),

$$Y = BXC^{-1} = A^{-1}* B^{-1}* C^{-1}$$

from which

$$B(A*B*C) = (A^{-1}* B^{-1}* C^{-1})C \qquad (99)$$

If $Y = -(A*B*C)$,

$$A(-Y + I) = B(-Y) + C \qquad \text{or} \qquad \overline{A}(Y - I) = \overline{B}Y + C$$

Postmultiplying by (0,-1) yields

$$\overline{A}(\overline{Y} + I) = \overline{B}\ \overline{Y} + \overline{C}$$

from which

$$\overline{Y} = \overline{A} * \overline{B} * \overline{C}$$

or

$$Y = -(A*B*C) = \overline{\overline{A} * \overline{B} * \overline{C}} \tag{100}$$

If A, B, or C is first order, trivolution can be expressed in terms of the basic operations. For example, if $A = (a_1, a_2)$, from Eq. (68)

$$A*B*C = (A - B)^{-1} (C - a_2) \tag{101}$$

Trivolution would not be a very valuable operation if it could be used to solve only one form of equation. Fortunately, in combination with the basic operations, trivolution can be used to solve many other types of equations. To solve

$$A(X - I) = BX + C \tag{102}$$

postmultiply both sides by (0,-1) to obtain

$$A(\overline{X} + I) = B\overline{X} + \overline{C}$$

from which

$$X = \overline{A * B * \overline{C}} \tag{103}$$

To solve

$$A(DX + E) = BX + C \tag{104}$$

postmultiply both sides by E^{-1} and replace B with $BD^{-1}D$, to obtain

$$A(DXE^{-1} + I) = BD^{-1} (DXE^{-1}) + CE^{-1}$$

from which

$$X = D^{-1}(A * BD^{-1} * CE^{-1})E \qquad (105)$$

To solve

$$A(X + D) = B(X + E) + C$$

we first rewrite the equation in the form

$$A[(X + E) + (D - E)] = B(X + E) + C \qquad (106)$$

Since Eq. (106) has the same form as Eq. (104), we can solve for X + E by Eq. (105), which yields

$$X = [A * B * C(D - E)^{-1}] (D - E) - E \qquad (107)$$

Other equations can be solved by using trivolution two or more times. As an example, we will solve

$$B[A(X + I) - E + X] = C[A(X + I) - E] + DX$$

for X. Postmultiplying by X^{-1} yields

$$B[A(I + X^{-1}) - EX^{-1} + I] = C[A(I + X^{-1}) - EX^{-1}] + D$$

Applying trivolution once, we obtain

$$A(I + X^{-1}) - EX^{-1} = B*C*D$$

Postmultiplying by X and adding E to both sides yields

$$A(X + I) = (B*C*D)X + E$$

from which

$$X = A * (B*C*D) * E$$

Similarly, $X = (A*B*C) * D * E$ is the solution of

$$A[(D + I)X + E + I] = B(DX + E) + C(X + I)$$

Unfortunately, there are many PWL operator equations which cannot be solved by using trivolution in combination with the basic algebraic operations. Such equations can often be solved using a combination of trivolution with an iterative procedure. As will be shown in Section V. G., sets of simultaneous equations which occur in the analysis of resistive PWL networks can be solved by using trivolution and iteration.

V. ANALYSIS OF RESISTIVE PWL NETWORKS

PWL operators have been defined to represent the characteristic curves of PWL elements, algebraic operations have been defined for PWL operators, the algebraic properties of these operations have been studied, and the basic techniques for solving PWL-operator equations have been discussed. We are now ready to apply these results to the analysis of resistive PWL networks. Methods for obtaining the input and transfer characteristics of series-parallel PWL networks and more general PWL networks containing one, two, or more PWL resistors will be studied.

A. SERIES-PARALLEL NETWORKS

The v-i characteristic of n PWL resistors in series is

$$R = \sum_{k=1}^{n} R_k \tag{108}$$

where R_k is the v-i characteristic of the kth resistor. By duality, the i-v characteristic of n PWL resistors in parallel is

$$G = \sum_{k=1}^{n} G_k \tag{109}$$

where G_k is the i-v characteristic of the kth resistor. Taking inverses, the v-i characteristic of n PWL resistors in parallel is given by

$$R = \left(\sum_{k=1}^{n} R_k^{-1} \right)^{-1} \tag{110}$$

The input v-i characteristic of any series-parallel network that is composed of PWL resistors can be found by using the above equations and inversion. As an example, the series-parallel network of Fig. 5a will be analyzed by PWL operators. The i-v characteristic to the right of a-a' is given by

$$\left[(-1,\ 1/2) + \begin{pmatrix} 0, & 0 \\ 6, & 3/2 \end{pmatrix} \right]^{-1} = \begin{pmatrix} -1, & 1/2 \\ 5, & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2, & 2 \\ -5/2, & 1/2 \end{pmatrix}$$

The v-i characteristic to the right of b-b' is

$$\left[\begin{pmatrix} 2, & 2 \\ -5/2, 1/2 \end{pmatrix} -3 + \begin{pmatrix} 0, & 0 \\ 1/2, 1/2 \end{pmatrix} -1 + \begin{pmatrix} 0, & 0 \\ -6, & 2 \end{pmatrix} 3 \right]^{-1} = \begin{pmatrix} 2, & 2 \\ -5/2, 1/2 \\ -2, & 1 \\ -8, & 3 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -1, 1/2 \\ 5, & 2 \\ 2, & 1 \\ 8/3, 1/3 \end{pmatrix}$$

The input v-i characteristic is then

$$\begin{pmatrix} -1, 1/2 \\ 5, & 2 \\ 2, & 1 \\ 8/3, 1/3 \end{pmatrix} \begin{matrix} -4 \\ -3 \\ 1 \end{matrix} + \begin{pmatrix} 0, & 0 \\ -8/3, 2/3 \end{pmatrix} 4 = \begin{pmatrix} -1, 1/2 \\ 5, & 2 \\ 2, & 1 \\ 8/3, 1/3 \\ 0, & 1 \end{pmatrix}$$

which checks Eq. (5).

The PWL characteristic of the network of Fig. 6c can easily be derived in terms of PWL operators. The i-v characteristic of Fig. 6a is

$$i_1 = \left[-3 + \begin{pmatrix} 6, 6 \\ 0, 0 \end{pmatrix} + \begin{pmatrix} 0, 0 \\ -6, 6 \end{pmatrix} \right] (v_1) = \begin{pmatrix} 6, & 3 \\ 0, -3 \\ -6, & 3 \end{pmatrix} (v_1) = A(v_1)$$

The terminal current and voltage in Fig. 6c are related to i_1 and v_1 by

$$i_2 = i_1 - v_1 = (A - I)(v_1)$$

$$v_2 = i_1 + v_1 = (A + I)(v_1)$$

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Solving the second equation for v_1 and substituting in the first yields

$$i_2 = (A - I)(A + I)^{-1}(v_2)$$

$$= \begin{pmatrix} 6, & 2 \\ 0, & -4 \\ -6, & 2 \end{pmatrix} \begin{pmatrix} 6, & 4 \\ 0, & -2 \\ -6, & 4 \end{pmatrix}^{-1} (v_2) = \begin{pmatrix} 6, & 2 \\ 0, & -4 \\ -6, & 2 \end{pmatrix} \begin{matrix} -1 \\ 1 \end{matrix} \begin{matrix} -1 \\ 1 \end{matrix} \begin{pmatrix} -3/2, & 1/4 \\ 0, & -1/2 \\ 3/2, & 1/4 \end{pmatrix} (v_2)$$

Carrying out the indicated multiplication, we obtain

$$i_2 = \begin{pmatrix} 3, 1/2 \\ 0, \ 2 \\ -3, 1/2 \end{pmatrix} (v_2)$$

which checks the result given in Fig. 6d.

Many of the simple rules for working with linear resistive series-parallel networks have PWL analogs. For the voltage divider of Fig. 34, the input voltage is

$$v = v_1 + v_2 = R_1 (i) + R_2 (i) = (R_1 + R_2) (i)$$

Solving for i yields

$$i = (R_1 + R_2)^{-1} (v)$$

The output voltage is

$$v_2 = R_2 (i) = R_2 (R_1 + R_2)^{-1} (v) \tag{111}$$

If R_1 and R_2 are linear, this equation reduces to the familiar voltage-divider rule

$$v_2 = \frac{r_2}{r_1 + r_2} v$$

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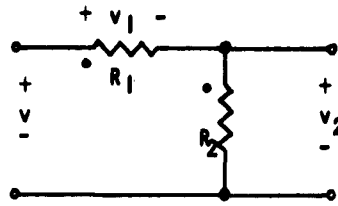
FIG. 34. PWL VOLTAGE DIVIDER.

For computational purposes, the transfer characteristic for the voltage divider of Fig. 34 can be rewritten as

$$R_2(R_1 + R_2)^{-1} = [(R_1 + R_2)R_2^{-1}]^{-1} = (R_1R_2^{-1} + I)^{-1} \qquad (112)$$

Given numerical values for R_1 and R_2, the latter expression is easy to evaluate. For

$$R_1 = \begin{pmatrix} -1, 1/2 \\ -1, \ 1 \\ 1, \ 0 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 0, \ 1 \\ -2, \ 2 \\ 2, \ 1 \end{pmatrix}$$

we obtain

$$R_1R_2^{-1} = \begin{pmatrix} -1, 1/2 \\ -1, \ 1 \\ 1, \ 0 \end{pmatrix} \begin{matrix} 0 \ \ 2 \\ 2 \ \ 4 \end{matrix} \begin{pmatrix} 0, \ 1 \\ 1, 1/2 \\ -2, \ 1 \end{pmatrix} = \begin{pmatrix} -1, 1/2 \\ -1, \ 1 \\ 1, \ 0 \\ -1, \ 0 \end{pmatrix}$$

and

$$(R_1R_2^{-1} + I) = \begin{pmatrix} -1, 3/2 \\ -1, \ 2 \\ 1, \ 1 \end{pmatrix} \quad (R_1R_2^{-1} + I)^{-1} = \begin{pmatrix} 2/3, 2/3 \\ 1/2, 1/2 \\ -1, \ 1 \end{pmatrix}$$

B. INPUT AND TRANSFER CHARACTERISTICS OF PWL LADDER NETWORKS

The input and transfer characteristics of PWL ladder networks can be found by applying standard ladder-network techniques. Consider the general PWL resistive ladder network of Fig. 35 with the v-i characteristics of the series branches represented by the PWL operators R_1, R_2, ..., R_n, and the i-v characteristics of the shunt branches represented by G_1, G_2, ..., G_n. The analysis proceeds as follows:

$$i_1 = G_1 \, (v_o)$$

$$v_1 = R_1 \, (i_1) + v_o = (R_1 G_1 + I) \, (v_o)$$

$$i_2 = G_2 \, (v_1) + i_1 = [G_2(R_1 G_1 + I) + G_1] \, (v_o)$$

$$v_2 = R_2 \, (i_2) + v_1 = \left\{ R_2[G_2(R_1 G_1 + I) + G_1] + R_1 G_1 + I \right\} \, (v_o)$$

and so on.

At the $(n-1)^{th}$ step, we have expressed i_{n-1} and v_{n-1} in terms of v_o. If $i_{n-1} = A(v_o)$ and $v_{n-1} = B(v_o)$, then the nth step is

$$i_n = G_n \, (v_{n-1}) + i_{n-1} = (G_n B + A) \, (v_o) \qquad (113)$$

$$v_n = R_n \, (i_n) + v_{n-1} = [R_n(G_n B + A) + B] \, (v_o) \qquad (114)$$

which expresses the transfer characteristic of the ladder. Eliminating
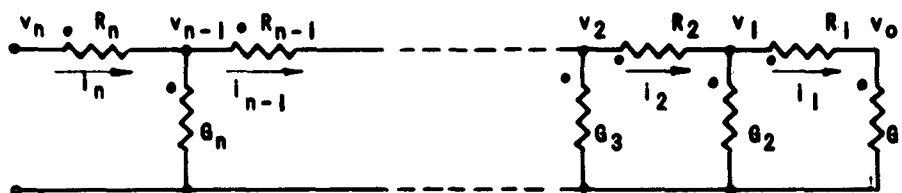
FIG. 35. GENERAL PWL LADDER NETWORK

v_o from Eqs.(113) and (114) gives the input v-i characteristic as

$$v_n = [R_n (G_n B + A) + B] (G_n B + A)^{-1} (i_n)$$

$$= [R_n + B (G_n B + A)^{-1}] (i_n) = [R_n + (G_n + AB^{-1})^{-1}] (i_n)$$

$$(115)$$

The above analysis shows that the input and transfer characteristics of any resistive PWL ladder network can be calculated in terms of PWL operators.

C. THE BRIDGE NETWORK

Non-series-parallel PWL networks are generally more difficult to analyze than series-parallel networks. The bridge (Fig. 36), which is the simplest non-series-parallel network, will be analyzed for various combinations of linear and PWL resistors. When the bridge is balanced, $i_5 = 0$, and $v_1 = v_2$. Using the voltage divider formula, Eq. (112),

$$v_1 = (R_1 R_2^{-1} + I)^{-1} (v)$$

and

$$v_2 = (R_3 R_4^{-1} + I)^{-1} (v)$$

Equating v_1 and v_2 gives the equation of balance as

$$(R_1 R_2^{-1} + I)^{-1} (v) = (R_3 R_4^{-1} + I)^{-1} (v)$$

or

$$R_1 R_2^{-1} = R_3 R_4^{-1}$$

The unbalanced bridge is easy to solve in certain cases. If R_5 and two adjacent arms are linear, a delta-wye or wye-delta transformation can be performed on the <u>linear</u> part of the network to reduce the bridge to a series-parallel network. If R_5 and an adjacent arm are PWL, such a
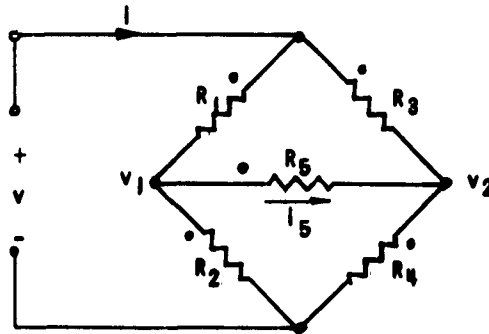
FIG. 36.   A PWL BRIDGE NETWORK.

transformation is impossible and it is necessary to write loop or node equations in order to analyze the network. The v-i characteristic of a diode bridge network[*] (Fig. 37) will be derived as an example.
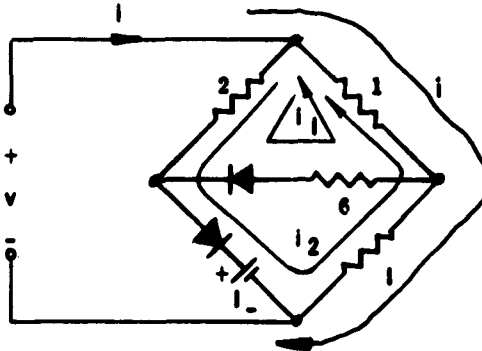
FIG. 37.   A DIODE BRIDGE NETWORK.

The equations for the indicated loops[**] are

$$(R_1 + 3) \, (i_1) + 3i_2 - i = 0 \tag{116}$$

$$3i_1 + (R_2 + 4) \, (i_2) - 2i = 0 \tag{117}$$

where

$$R_1 = \begin{pmatrix} 0, & 6 \\ 0, & \infty \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 1, & \infty \\ 1, & 0 \end{pmatrix}$$

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[*] Stern [Ref. 5, pp. 20-23] analyzes this same network using ϕ^+ transformations.

[**]The loops have been chosen so that each PWL resistor is traversed by a single loop current.

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The input voltage is given by

$$v = 2i - i_1 - 2i_2 = R(i) \tag{118}$$

If we substitute $X(i)$ for i_1 and $Y(i)$ for i_2 in Eqs. (116), (117), and (118), and then cancel i, we obtain three simultaneous PWL operator equations:

$$(R_1 + 3)X + 3Y - I = 0 \tag{119}$$

$$3X + (R_2 + 4)Y - 2I = 0 \tag{120}$$

$$R = 2I - X - 2Y \tag{121}$$

To solve these equations for R, we first use Eq. (121) to eliminate Y from Eq. (119) and X from Eq. (120), obtaining

$$(R_1 + 3/2)X + 2I - (3/2)R = 0 \quad \text{or} \quad (2R_1 + 3)X = 3R - 4$$

$$(R_2 - 2)Y + 4I - 3R = 0 \quad \text{or} \quad (R_2 - 2)Y = 3R - 4$$

Next, we solve these equations for X and Y respectively and substitute into Eq. (121) to obtain

$$(R - 2) + (2R_1 + 3)^{-1}(3R - 4) + 2(R_2 - 2)^{-1}(3R - 4) = 0 \tag{122}$$

Observing that

$$(R - 2) = 1/3 \, (3R - 4) - 2/3$$

and applying the right distributive law, Eq. (122) can be rewritten as

$$[1/3 + (2R_1 + 3)^{-1} + 2(R_2 - 2)^{-1}] \, (3R - 4) = 2/3$$

Solving this equation for R, we obtain

$$R = 1/3 \left\{ [1/3 + (2R_1 + 3)^{-1} + 2(R_2 - 2)^{-1}]^{-1} \cdot 2/3 + 4 \right\} \qquad (123)$$

This equation can be evaluated for the given values of R_1 and R_2 as follows:

$$R = 1/3 \left\{ \left[1/3 + \begin{pmatrix} 0, & 15 \\ 0, & \infty \end{pmatrix}^{-1} + 2 \begin{pmatrix} 1, & \infty \\ 1, & -2 \end{pmatrix}^{-1} \right]^{-1} \cdot 2/3 + 4 \right\}$$

$$= 1/3 \left\{ \left[1/3 + \begin{pmatrix} 0, 1/15 \\ 0, & 0 \end{pmatrix} 0 + \begin{pmatrix} 0, & 0 \\ 1, & -1 \end{pmatrix} 1 \right]^{-1} \cdot 2/3 + 4 \right\}$$

$$= 1/3 \left[\begin{pmatrix} 0, & 2/5 \\ 0, & 1/3 \\ 1, & -2/3 \\ 1, & -3/5 \end{pmatrix}^{-1} \cdot 2/3 + 4 \right] = 1/3 \left[\begin{pmatrix} 0, & 5/2 \\ 0, & 3 \\ 3/2, & -3/2 \\ 5/3, & -5/3 \end{pmatrix} \cdot 2/3 + 4 \right]$$

$$= 1/3 \begin{pmatrix} 0, & 17/3 \\ 0, & 6 \\ 3/2, & 3 \\ 5/3, & 26/9 \end{pmatrix} = \begin{pmatrix} 0, & 17/9 \\ 0, & 2 \\ 1/2, & 1 \\ 5/9, & 26/27 \end{pmatrix}$$

The resulting PWL operator represents the input v-i characteristic of the diode bridge network.

Figure 38a shows a bridge network with three PWL elements. To facilitate analysis of this network, the voltage source is replaced with two parallel sources (Fig. 38b), and then these sources are transformed to current sources (Fig. 38c). In the final network, R_2' is the parallel combination of r_1 and R_2, and R_4' is the parallel combination of r_3 and R_4. The voltage around the loop is

$$R_5 (i_5) + R_4' [(v/r_3) + i_5] - R_2' [(v/r_1) - i_5] = 0$$

When this equation is solved to obtain the relation between i_5 and v, the other voltages and currents in the network can be expressed as

functions of v. Substituting $Y(i_5)$ for v, we obtain the PWL operator equation

$$R_4' \left(\frac{1}{r_3} Y + I\right) = R_2' \left(\frac{1}{r_1} Y - I\right) - R_5$$

The change of variable, $Y = (r_1 + r_3)X + r_1$, reduces this to the form

$$R_4' \left(1 + \frac{r_1}{r_3}\right) (X + I) = R_2' \left(1 + \frac{r_3}{r_1}\right) X - R_5$$

which can be solved for X by trivolution. An example of the solution of a bridge network with four PWL elements is given in Section V. H.

D. RESISTIVE PWL NETWORK WITH ONE PWL RESISTOR

In the preceding sections, we have seen that it is always possible to analyze series-parallel PWL networks in terms of PWL operators and that solutions can also be obtained for simple non-series-parallel networks. We will now try to determine what class of resistive PWL networks can be analyzed by PWL operators and formulate a general method of analysis. We will start by deriving the input v-i characteristic for a linear resistive network that contains one PWL resistor imbedded in it. To facilitate the analysis, the linear part of the network is separated from the PWL part, and the network is redrawn as a linear two-port terminated in a PWL resistor (Fig. 39).
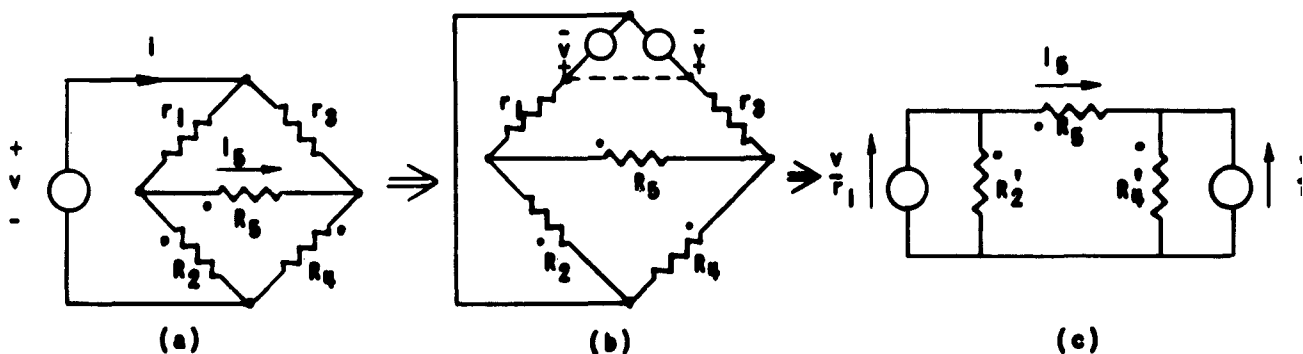
FIG. 39. A BRIDGE WITH THREE PWL RESISTORS.

FIG. 39. GENERAL RESISTIVE NETWORK WITH
ONE PWL RESISTOR.

This procedure permits part of the work to be done on the linear portion before the nonlinear portion is considered. The two-port may contain dependent sources and non-reciprocal elements just as long as it is linear. The two-port can be described by its z-parameters, and the terminal behavior can be expressed by the equations[*]

$$v_1 = z_{11}i_1 - z_{12}i_2 \tag{124}$$

$$v_2 = z_{21}i_1 - z_{22}i_2 = R(i_2) \tag{125}$$

Solving the second equation for i_2 and substituting in the first equation, we obtain

$$i_2 = (R_2 + z_{22})^{-1} z_{21}(i_1) \tag{126}$$

$$v_1 = z_{11}i_1 - z_{12}(R_2 + z_{22})^{-1} z_{21} (i_1)$$

$$= [z_{11} - z_{12} (R_2 + z_{22})^{-1} z_{21}] (i_1) \tag{127}$$

As a partial check on this result, note that when R_2 is linear the above equation reduces to the correct form

$$v_1 = \left(z_{11} - \frac{z_{12}z_{21}}{r_2 + z_{22}} \right) i_1$$

---

[*] The minus signs result from the choice of reference directions for the currents.

As an example, we will apply Eq. (127) to determine the input characteristic for the network of Fig. 40. For this network,

$$z_{11} = 3, \qquad z_{12} = z_{21} = 1, \qquad z_{22} = 2, \qquad R = \begin{pmatrix} 0, & 1 \\ 1, & 1/2 \end{pmatrix}$$

$$(R + z_{22}) = \begin{pmatrix} 0, & 3 \\ 1, & 5/2 \end{pmatrix} \qquad (R + z_{22})^{-1} = \begin{pmatrix} 0, & 1/3 \\ -2/5, & 2/5 \end{pmatrix}$$

$$z_{11} - z_{12} (R_2 + z_{22})^{-1} z_{21} = 3 - \begin{pmatrix} 0, & 1/3 \\ -2/5, & 2/5 \end{pmatrix} = \begin{pmatrix} 0, & 8/3 \\ 2/5, 13/5 \end{pmatrix}$$

This same result could also be obtained by a series-parallel analysis.

The transfer characteristic that relates any two voltages in a linear resistive network with one PWL resistor can also be found. If we add a third port to the linear network of Fig. 39 and set $i_3 = 0$, the terminal behavior of the network is now described by Eq. (124), Eq. (125), and

$$v_3 = z_{31} i_1 - z_{32} i_2 \tag{128}$$

From Eq. (127),

$$i_1 = [z_{11} - z_{12} (R_2 + z_{22})^{-1} z_{21}]^{-1} (v_1) \tag{129}$$

Substituting Eqs. (126) and (129) into Eq. (128), we obtain the desired transfer characteristic as

$$v_3 = [z_{31} - z_{32} (R_2 + z_{22})^{-1} z_{31}] [z_{11} - z_{12} (R_2 + z_{22})^{-1} z_{21}]^{-1}(v_1) \tag{130}$$
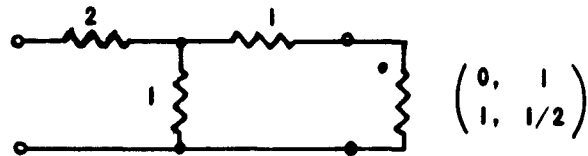
FIG. 40. EXAMPLE OF A NETWORK WITH ONE PWL RESISTOR.

E. PWL NETWORK WITH TWO PWL RESISTORS

A linear resistive network with two PWL resistors imbedded in it can be analyzed in a manner similar to that with only one PWL resistor. The network is first redrawn as a linear three-port terminated in two PWL resistors (Fig. 41) and the terminal relations for the network are expressed in terms of the z-parameters by the equations:

$$v_1 = z_{11} \, i_1 - z_{12} \, i_2 - z_{13} i_3 \tag{131}$$

$$v_2 = z_{21} \, i_1 - z_{22} \, i_2 - z_{23} \, i_3 = R_2 \, (i_2) \tag{132}$$

$$v_3 = z_{31} \, i_1 - z_{32} \, i_2 - z_{33} \, i_3 = R_3 \, (i_3) \tag{133}$$

Combining the last two equations gives

$$z_{21} \, z_{31} \, i_1 = z_{31} \, (R_2 + z_{22}) \, (i_2) + z_{31} \, z_{23} \, i_3$$

$$= z_{21} \, (R_3 + z_{33}) \, (i_3) + z_{21} \, z_{32} \, i_2$$

Solving this equation for i_3 yields

$$i_3 = [z_{21} \, (R_3 + z_{33}) - z_{31}z_{23}]^{-1} \, [z_{31}(R_2 + z_{22}) - z_{21} \, z_{32}] \, (i_2)$$

$$= A \, (i_2)$$

Substituting A (i_2) for i_3 in Eq. (132) and A^{-1} (i_3) for i_2 in Eq. (133) and solving, we obtain

$$i_2 = (R_2 + z_{22} + z_{23} \, A)^{-1} \, z_{21}(i_1)$$

$$i_3 = (R_3 + z_{33} + z_{32} \, A^{-1})^{-1} \, z_{31} \, (i_1)$$

from which

$$v_1 = [z_{11} - z_{12}(R_2 + z_{22} + z_{23}A)^{-1}z_{21}$$

$$- z_{13}(R_3 + z_{33} + z_{32}A^{-1})^{-1}z_{31}](i_1) \qquad (134)$$

where

$$A = [z_{21}(R_3 + z_{33}) - z_{31}z_{23}]^{-1}[z_{31}(R_2 + z_{22}) - z_{21}z_{32}]$$
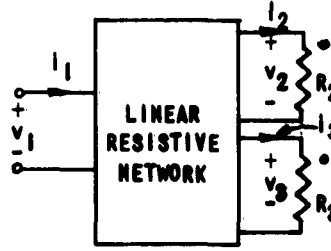
FIG. 41. GENERAL RESISTIVE NETWORK WITH TWO PWL RESISTORS.

Evaluation of this expression for given values of the constants may be vairly laborious. In the special case where $z_{12}z_{31}/z_{23} = z_{13}z_{21}/z_{32}$, which includes all reciprocal networks, a simpler equation for the PWL input impedance can be derived. Following a procedure similar to that used to solve the diode bridge network (Fig. 37), we can show that

$$R = \left[z_{21}\left(R_2 + z_{22} - \frac{z_{12}\,z_{23}}{z_{13}}\right)^{-1} \frac{z_{23}}{z_{13}} + z_{13}\left(R_3 + z_{33} - \frac{z_{13}\,z_{32}}{z_{12}}\right)^{-1} \frac{z_{32}}{z_{12}}\right.$$

$$\left. + I\right]^{-1} \frac{z_{13}\,z_{21}}{z_{23}} + \left(z_{11} - \frac{z_{13}\,z_{21}}{z_{23}}\right) \qquad (135)$$

Evaluation of this expression requires only one addition of two higher-order PWL operators compared with three additions and one multiplication for Eq. (134). Voltage-transfer characteristics for the resistive network with two PWL resistors can also be found by an extension of the procedure used in Section V. D.

F. PWL NETWORK WITH THREE PWL RESISTORS

In Sects. V. D. and V. E., general methods of analyzing resistive PWL networks that contain one or two PWL resistors have been derived. If possible, we would like to generalize this procedure to three or more PWL resistors. Unfortunately, attempts to do this lead to PWL operator equations that cannot be solved in terms of the basic operations of addition, subtraction, multiplication, and inversion, so trivolution and iterative procedures must be used.

For purposes of determining the PWL input characteristic, a linear resistive network that contains n PWL resistors imbedded in it can be redrawn as a linear (n+1)-port terminated in n PWL resistors (Fig. 42).
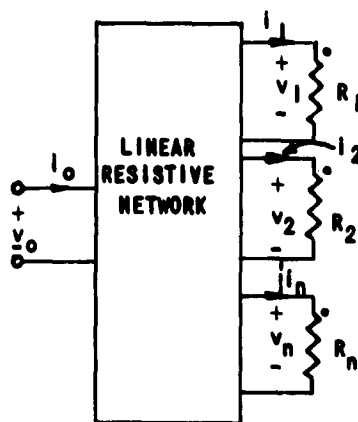
FIG. 42. GENERAL RESISTIVE NETWORK WITH n PWL RESISTORS.

As long as it is linear, the (n+1)-port may contain dependent sources or other non-reciprocal elements. When d-c sources are present in the (n+1)-port, equivalent sources may be brought out at the terminals and combined with the PWL elements.

Instead of describing the terminal characteristics of the (n+1)-port by z-parameters that relate the v's to the i's, it is more convenient to choose the voltages and currents at some of the ports as independent variables and express the remaining voltages and currents in terms of these variables. For n = 3, we choose v_2, i_2, v_3, and i_3 as independent variables and then express v_0, i_0, v_1, and i_1 in terms of these variables by the equations.

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$$v_0 = a_{02} \, v_2 + a_{03} \, v_3 + b_{02} \, i_2 + b_{03} \, i_3$$

$$v_1 = a_{12} \, v_2 + a_{13} \, v_3 + b_{12} \, i_2 + b_{13} \, i_3 \qquad (136)$$

$$i_0 = c_{02} \, v_2 + c_{03} \, v_3 + d_{02} \, i_2 + d_{03} \, i_3$$

$$i_1 = c_{12} \, v_2 + c_{13} \, v_3 + d_{12} \, i_2 + d_{13} \, i_3$$

The coefficients are, in a sense, a generalization of the ABCD-parameters that are used to describe two-ports. These coefficients can be calculated directly from the linear network or expressed in terms of the z-parameters.

We will now formulate the equations for finding the input PWL characteristic of a linear four-port terminated in three PWL resistors. We can eliminate the voltages from Eq. (136) by the substitutions

$$v_0 = R_0 \, (i_0), \qquad v_1 = R_1 \, (i_1), \qquad v_2 = R_2 \, (i_2), \quad \text{and} \quad v_3 = R_3 \, (i_3),$$

where R_1, R_2, and R_3 are the impedances of the three PWL resistors, and R_0 is the desired input impedance. Performing these substitutions, we obtain

$$v_0 = (a_{02} \, R_2 + b_{02}) \, (i_2) + (a_{03} \, R_3 + b_{03}) \, (i_3)$$

$$\qquad (137)$$

$$= R_0[(c_{02} \, R_2 + d_{02}) \, (i_2) + (c_{03} \, R_3 + d_{03}) \, (i_3)]$$

$$v_1 = (a_{12} \, R_2 + b_{12}) \, (i_2) + (a_{13} \, R_3 + b_{13}) \, (i_3)$$

$$\qquad (138)$$

$$= R_1[(c_{12} \, R_2 + d_{12}) \, (i_2) + (c_{13} \, R_3 + d_{13}) \, (i_3)]$$

To simplify these equations, we let $i_3 = Q(i_2)$ and

$$\left. \begin{array}{l} R_{jk} = a_{jk} R_k + b_{jk} \\ S_{jk} = c_{jk} R_k + d_{jk} \end{array} \right\} \qquad (j = 0,1; \quad k = 2,3) \qquad (139)$$

After performing these substitutions and cancelling i_2, we obtain

$$R_0 (S_{02} + S_{03} Q) = R_{02} + R_{03} Q \qquad (140)$$

$$R_1 (S_{12} + S_{13} Q) = R_{12} + R_{13} Q \qquad (141)$$

Using Eq. (105), we can express the solution to Eq. (141) in terms of trivolution as

$$Q = S_{13}^{-1} (R_1 * R_{13} S_{13}^{-1} * R_{12} S_{12}^{-1}) S_{12} \qquad (142)$$

From Eq. (140), the desired input impedance is

$$R_0 = (R_{02} + R_{03} Q) (S_{02} + S_{03} Q)^{-1} \qquad (143)$$

Thus, by using trivolution in combination with the basic operations, we can find the PWL input impedance of any resistive network that contains three PWL resistors.

G. RESISTIVE PWL NETWORK WITH N PWL RESISTORS

When n is odd, the linear resistive network with n PWL resistors (Fig. 42) can be solved by an extension of the procedure used for n = 3. If m = (n-1)/2, we choose the last m+1 voltages and currents as independent variables and express the other voltages and currents by the matrix equation

$$
\begin{bmatrix}
v_0 \\
\vdots \\
v_m \\
\hline
i_0 \\
\vdots \\
i_m
\end{bmatrix}
=
\left[
\begin{array}{ccc|ccc}
a_{00} & \cdots & a_{0m} & b_{00} & \cdots & b_{0m} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{m0} & \cdots & a_{mm} & b_{m0} & \cdots & b_{mm} \\
\hline
c_{00} & \cdots & c_{0m} & d_{00} & \cdots & d_{0m} \\
\vdots & & \vdots & \vdots & & \vdots \\
c_{m0} & \cdots & c_{mm} & d_{m0} & \cdots & d_{mm}
\end{array}
\right]
\begin{bmatrix}
v_0' \\
\vdots \\
v_m' \\
\hline
i_0' \\
\vdots \\
i_m'
\end{bmatrix}
\qquad (144)
$$

where $v_k' = v_{k+m+1}$ and $i_k' = i_{k+m+1}$ \qquad (k = 0,1,...,m)

The sub-matrices $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$, and $[d_{ij}]$ can be determined from the impedance matrix of the linear network.*

The matrix equation represents n linear equations. We eliminate the voltages from these equations by the substitutions

$$v_j = R_j(i_j) \qquad (j = 0,1,\ldots,m)$$

$$v'_k = R'_k(i'_k) = R_{k+m+1}(i'_k) \qquad (k = 0,1,\ldots,m)$$

to obtain m+1 equations of the form

$$v_j = \sum_{k=0}^{m} R_{jk}(i'_k) = R_j[\sum_{k=0}^{m} S_{jk}(i'_k)] \qquad (j = 0,1,\ldots,m) \qquad (145)$$

where

$$\left.\begin{array}{l} R_{jk} = a_{jk}R'_k + b_{jk} \\[2mm] S_{jk} = c_{jk}R'_k + d_{jk} \end{array}\right\} \qquad (j = 0,1,\ldots,m;\ k = 0,1,\ldots,m) \qquad (146)$$

If we make the substitutions

$$i'_k = X_k(i'_0) \qquad (k = 0,1,\ldots,m)$$

and then cancel i'_0, we obtain m+1 PWL operator equations of the form

$$\sum_{k=0}^{m} R_{jk}X_k = R_j\,(\sum_{k=0}^{m} S_{jk}X_k) \qquad (j = 0,1,\ldots,m) \qquad (147)$$

where $X_0 = I$ and the remaining X_k's are unknown PWL operators. Solving the first equation (j = 0) for R_0, we obtain

---

* Except for some changes in sign, the matrix in Eq. (144) is the same as the transmission or chain matrix of the network. The relation between the transmission matrix and the impedance matrix is discussed in Bayard [Ref. 7].

$$R_o = \left(\sum_{k=0}^{m} R_{ok} X_k \right) \left(\sum_{k=0}^{m} S_{ok} X_k \right)^{-1} \qquad (148)$$

The m unknown X_k's can be found by solving the remaining m equations simultaneously by using iteration and trivolution. Rewriting Eq. (147) in the form

$$R_j \left(S_{jj} X_j + \sum_{\substack{k=0 \\ (k \neq j)}}^{m} S_{jk} X_k \right) = R_{jj} X_j + \sum_{\substack{k=0 \\ (k \neq j)}}^{m} R_{jk} X_k \qquad (j = 1,2,\ldots,m) \qquad (149)$$

we can use Eq. (105) to solve for X_j. We have now demonstrated that the jth equation can be solved for X_j in terms of the other m-1 X_k's. Thus, we have reduced the problem to solving a set of simultaneous equations of the form

$$X_j = f_j (X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_m) \qquad (j = 1,2,\ldots,m) \qquad (150)$$

To solve these equations, we can set up an iteration of the form

$$X_j^{k+1} = f_j (X_1^k, X_2^k, \ldots, X_{j-1}^k, X_{j+1}^k, \ldots, X_m^k) \qquad (j = 1,2,\ldots,m) \qquad (151)$$

where X_j^k is the kth approximation to X_j.

A general procedure for deriving the input v-i characteristic of a resistive PWL network can now be stated:

1. Whenever possible, select two-terminal sub-networks that contain PWL resistors and reduce each sub-network to a single equivalent PWL resistor.

2. When all such simplifications have been made, redraw the network as a linear n+1-port terminated in n PWL resistors.

3. If n is odd, determine the matrix of coefficients in Eq. (144) and then calculate the R_{jk}'s and S_{jk}'s by Eq. (146).

4. Write $m = (n-1)/2$ simultaneous equations of the form of Eq. (149).

5. Solve these equations using an iteration of the form of Eq. (151) and then calculate R_o by Eq. (148).

A similar procedure has been derived for n even. In this case, the problem reduces to the solution of n/2 simultaneous PWL-operator equations.

The derivation in this section serves to illustrate a general approach to the analysis of resistive PWL networks. We have shown that the analysis of a network with n PWL resistors can always be reduced to the solution of n/2 or fewer simultaneous PWL-operator equations. The general method outlined above is rather cumbersome to use and, for the solution of many PWL-network problems, it is easier to formulate a special procedure instead of using the method exactly as given above.

H. EXAMPLES OF ITERATIVE SOLUTIONS

The bridge network of Fig. 43, which has four PWL resistors, can be described by the loop equations

$$R_2(i_1) + R_1(i_1 + i) + (i_1 - i_2) = 0$$

$$R_3(i_2) + R_4(i_2 + i) + (i_2 - i_1) = 0$$

If we let $i_1 = X(i)$ and $i_2 = Y(i)$ and cancel i, we can rewrite these equations as

$$Y = (R_2 + I)X + R_1(X + I) \qquad (152)$$

$$X = (R_3 + I)Y + R_4(Y + I) \qquad (153)$$

If we eliminate Y by substituting the first equation into the second, we obtain a PWL-operator equation in X which we do not know how to solve, so we will solve Eqs. (152) and (153) with an iterative procedure. Using trivolution to solve Eq. (152) for X and Eq. (153) for Y yields

$$X = R_1 * -(R_2 + I) * Y$$

$$Y = R_4 * -(R_3 + I) * X$$
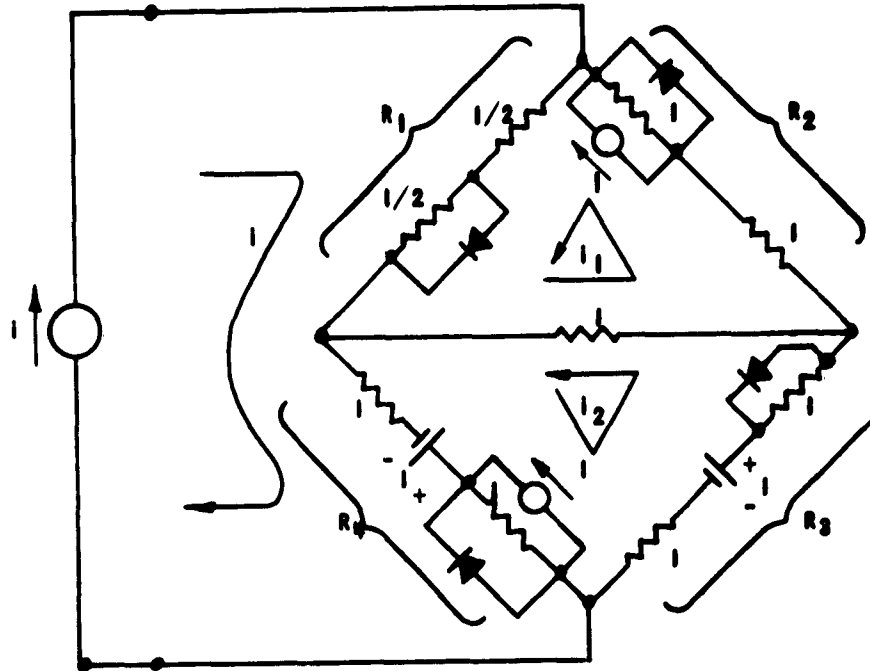
FIG. 43. BRIDGE NETWORK WITH FOUR PWL RESISTORS.

We then set up the iteration

$$X_{k+1} = R_1 * -(R_2 + I) * Y_k \qquad (154)$$

$$Y_{k+1} = R_4 * -(R_3 + I) * X_{k+1} \qquad (155)$$

where X_k is the kth approximation to X, and Y_k is the kth approximation to Y. It can be shown that a sufficient condition for the convergence of an iteration of the form

$$X_{k+1} = A * B * Y_k$$

$$Y_{k+1} = C * D * X_{k+1}$$

is $\qquad \rho = |r_{aj} - r_{bk}|_{min} \cdot |r_{cj} - r_{dk}|_{min} > 1$

where $|r_{aj} - r_{bk}|_{min}$ is the minimum value of $|r_{aj} - r_{bk}|$,

r_{aj} is the jth slope of A, and r_{bk} is the kth slope of B.

For the network of Fig. 43, we have

$$A = R_1 = \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix}$$

$$B = -(R_2 + I) = - \left[\begin{pmatrix} -1, & 2 \\ 0, & 1 \end{pmatrix} + (0,1) \right] = \begin{pmatrix} 1, & -3 \\ 0, & -2 \end{pmatrix}$$

$$C = R_4 = \begin{pmatrix} -1, & 1 \\ 0, & 2 \end{pmatrix}$$

$$D = -(R_3 + I) = - \left[\begin{pmatrix} -1, & 1 \\ -1, & 2 \end{pmatrix} + (0,1) \right] = \begin{pmatrix} 1, & -2 \\ 1, & -3 \end{pmatrix}$$

For these values, $\rho = 15/2$ and convergence is assured. The iteration was carried out on the Burroughs 220 Computer (see Section IX. D.). Starting with an initial approximation $Y_0 = 0$, the iteration converged to four-decimal-place accuracy in six iterations. The following solutions were obtained

$$X = \begin{bmatrix} -4.0000, & 2.0000 \\ -2.4286, & 1.2857 \\ -1.8571, & 1.0000 \\ -0.5000, & 0.5000 \\ 0.6000, & 0.2000 \\ 3.0556, & -0.5556 \end{bmatrix} \qquad Y = \begin{bmatrix} -4.0000, & 2.0000 \\ -2.4286, & 1.4286 \\ -1.8571, & 1.1429 \\ -0.5000, & 0.5000 \\ 0.6000, & 0.0000 \\ 4.0667, & -2.0000 \end{bmatrix}$$

The input PWL resistance to the network is

$$R_o = R_1(X + I) + R_4 (Y + I) = \begin{bmatrix} -4.0000, & -5.0000 \\ -2.4286, & -3.1429 \\ -1.8571, & -2.2857 \\ -0.5000, & 0.0000 \\ 0.6000, & 1.6000 \\ 4.0667, & 5.7333 \end{bmatrix} \tag{156}$$

As a second example, we will derive the input and transfer characteristics of the bridged-tee network of Fig. 44. The nodal equations for this network are

$$e_2 + G_o \ (e_2 - e_1) - G_2 \ (e_3 - e_2) = 0$$

$$G_m \ (e_3) + G_2 \ (e_3 - e_2) - G_1 \ (e_1 - e_3) = 0$$
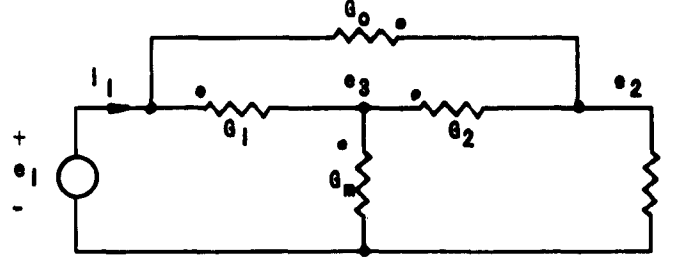
FIG. 44. BRIDGED-TEE NETWORK WITH FOUR PWL RESISTORS.

If we replace e_2 with $X(e_1)$ and e_3 with $Y(e_1)$ and then cancel e_1, we obtain the PWL-operator equations

$$X + G_o \ (X - I) - G_2(Y - X) = 0 \qquad (157)$$

$$G_m \ (Y) - G_1 \ (I - Y) + G_2 \ (Y - X) = 0 \qquad (158)$$

To solve Eq. (157) for X, we first rewrite it in the equivalent form

$$\overline{G}_2[(X - I) + (I - Y)] = G_o'(X - I) + I$$

where $G_o' = G_o + I$. Postmultiplying by $(I - Y)^{-1}$ and solving for $(X - I)$ by trivolution, we obtain

$$(X - I) = [\overline{G}_2 * G_o' * (I - Y)^{-1}] (I - Y) \qquad (159)$$

Adding Eqs. (157) and (158) and rearranging terms, we obtain

$$G_1 \ (-Y + I) = G_m \ (Y) + G_o \ (X - I) + X = \overline{G}_m \ (-Y) + G_o'X + I$$

Solving for -Y by trivolution yields

$$-Y = G_1 * \overline{G}_m * (G_o'X + I) \qquad (160)$$

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On the basis of Eqs. (159) and (160), we can set up the iteration

$$X_{k+1} = [\overline{G}_2 * G_o' * (I - Y_k)^{-1}] (I - Y_k) + I$$

$$Y_{k+1} = -[G_1 * \overline{G}_m * (G_o' X_{k+1} + I)]$$

This iteration was carried out on the Burroughs 220 Computer using the following values for the PWL conductances

$$G_1 = G_2 = \begin{bmatrix} 0, & -.1 \\ 1, & 0 \\ 1.1, & 1 \end{bmatrix} \qquad G_o = \begin{bmatrix} -1, & -.1 \\ 0, & 0 \\ 1, & 1 \end{bmatrix} \qquad G_m = \begin{bmatrix} -.1, & 0 \\ 0, & 1 \\ 1, 1.1 \end{bmatrix}$$

Starting with the initial approximation $X_o = (0, 1/2)$, the iteration converged to four-decimal-place accuracy in seven iterations. The following values were obtained

$$X = \begin{bmatrix} -3.6889, -1.3445 \\ -2.2626, -1.1313 \\ -0.1000, -0.1000 \\ 0.9107, -0.0149 \\ 1.0899, 1.0075 \\ 2.3232, 1.3322 \\ 3.5556, 2.4656 \end{bmatrix} \qquad Y = \begin{bmatrix} -3.1285, -0.2000 \\ -2.2626, -0.1313 \\ -0.1000, -0.1000 \\ 0.9107, 0.9256 \\ 1.0899, 1.0824 \\ 2.3232, 2.1121 \\ 3.5556, 2.3233 \end{bmatrix}$$

X represents the transfer characteristic between e_2 and e_1, and Y represents the transfer characteristic between e_3 and e_1. The input current to the network is

$$i_1 = [G_1 (I - Y) - G_o (X - I)] (e_1) = \begin{bmatrix} -3.6889, -2.7889 \\ -2.2626, -1.4444 \\ -0.1000, -0.0997 \\ 0.9107, 0.0926 \\ 1.0899, 1.0075 \\ 2.3232, 1.3322 \\ 3.5556, 2.4656 \end{bmatrix} (e_1)$$

Using techniques similar to those illustrated above, several other

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examples of PWL network equations were solved iteratively with the computer. In problems of this type, it may be necessary to try several different iteration schemes for solving the PWL-operator equations before one is found which converges satisfactorily.

VI. PWL TWO-PORTS AND PIECEWISE-PLANAR FUNCTIONS

In the preceding chapter, we considered methods for determining the input and transfer characteristics of resistive PWL networks driven by a single source. We will now consider the problem of describing the characteristics of a PWL two-port which may be driven by two independent sources. By analogy with linear networks, one might expect that a PWL two-port could be described by equations of the form

$$v_1 = R_{11} (i_1) + R_{12} (i_2)$$

$$v_2 = R_{21} (i_1) + R_{22} (i_2)$$

Unfortunately, equations of this form apply only in special cases. In general, the input voltages to a PWL two-port will be piecewise-planar functions of the input currents. That is, a three-dimensional plot of $v_1 = f(i_1,i_2)$ will consist of a series of planar sections that meet at breaklines. In each planar region, v_1 is a linear function of i_1 and i_2. Piecewise-planar functions can be described mathematically by specifying the equation of each plane together with a set of inequalities that describe the region in which this equation is valid.

A more convenient representation of piecewise-planar functions in terms of PWL operators has been studied, and attempts to generalize the concept of PWL operators to piecewise-planar operators have been made. Lattice, tee, and pi networks have been analyzed to provide examples of piecewise-planar functions.

A. PWL SYMMETRIC LATTICES

A symmetric lattice network composed of PWL resistors (Fig. 45a) is relatively easy to analyze. In Fig. 45b, the lattice is redrawn as a bridge with the current sources redistributed. It is easy to verify that the currents supplied to nodes a, b, c, and d by the current sources are the same in Figs. 45a and 45b, and therefore the voltages between the nodes are unchanged. From symmetry, the net current flowing around
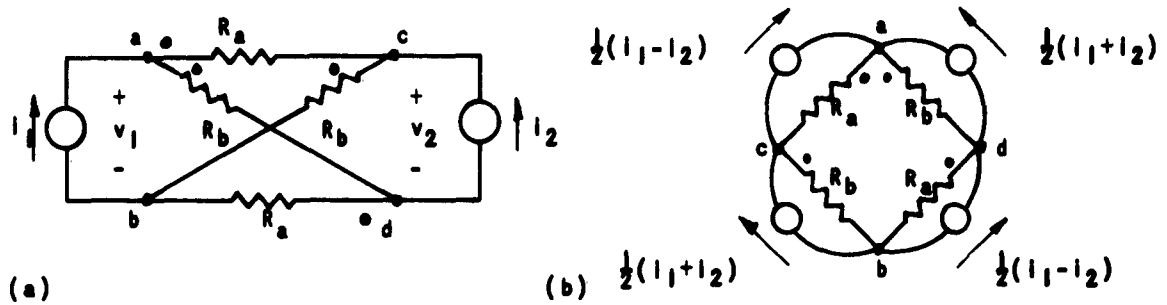
FIG. 45. SYMMETRIC PWL LATTICE.

the loop adbca in Fig. 45b is zero, so the terminal voltages are

$$v_{ab} = v_1 = R_b^{\frac{1}{2}} (i_1 + i_2) + R_a^{\frac{1}{2}} (i_1 - i_2) \qquad (161)$$

$$v_{cd} = v_2 = R_b^{\frac{1}{2}} (i_1 + i_2) - R_a^{\frac{1}{2}} (i_1 - i_2) \qquad (162)$$

By adding and subtracting these equations, we obtain

$$v_1 + v_2 = 2R_b^{\frac{1}{2}} (i_1 + i_2) \qquad \text{and} \qquad v_1 - v_2 = 2R_a^{\frac{1}{2}} (i_1 - i_2)$$

Solving these equations for i_1 and i_2, we obtain

$$i_1 = R_b^{-1\frac{1}{2}} (v_1 + v_2) + R_a^{-1\frac{1}{2}} (v_1 - v_2) \qquad (163)$$

$$i_2 = R_b^{-1\frac{1}{2}} (v_1 + v_2) - R_a^{-1\frac{1}{2}} (v_1 - v_2) \qquad (164)$$

Thus, for the PWL symmetric lattice, PWL operators can be used to express the v's in terms of the i's and conversely. The general PWL lattice is much more difficult to analyze.

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B. PWL TEE NETWORKS

Although it is easy to express the terminal voltages of a PWL T-network as a function of the terminal currents, it is often difficult to solve for the currents in terms of the voltages. For the network of Fig. 46,

$$v_1 = R_1 (i_1) + R_3 (i_1 + i_2) \qquad (165)$$

$$v_2 = R_2 (i_2) + R_3 (i_1 + i_2) \qquad (166)$$
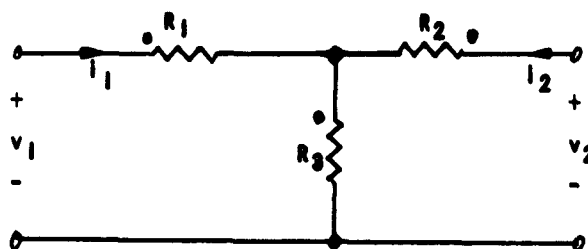
FIG. 46. PWL T-NETWORK.

When any two of the three PWL resistors are linear, it is possible to solve for i_1 and i_2. For example, if R_1 and R_2 are linear, the sum of Eq. (165) and the product of (r_1/r_2) and Eq. (166) is

$$v_1 + \frac{r_1}{r_2} v_2 = (R_3 + \frac{r_1}{r_2} R_3 + r_1)(i_1 + i_2) \qquad (167)$$

from which

$$i_1 + i_2 = (R_3 + \frac{r_1}{r_2} R_3 + r_1)^{-1} (v_1 + \frac{r_1}{r_2} v_2) \qquad (168)$$

$$= A (v_1 + \frac{r_1}{r_2} v_2)$$

Subtracting Eq. (166) from Eq. (165), we obtain

$$v_1 - v_2 = r_1 i_1 - r_2 i_2 \qquad (169)$$

Solving Eq. (167) and (169) simultaneously for i_1 and i_2 yields

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$$i_1 = \frac{1}{r_1 + r_2} \left[(v_1 - v_2) + r_2 \ A \ (v_1 + \frac{r_1}{r_2} v_2) \right] \qquad (170)$$

$$i_2 = \frac{1}{r_1 + r_2} \left[(v_2 - v_1) + r_1 \ A \ (v_1 + \frac{r_1}{r_2} v_2) \right] \qquad (171)$$

When only one of the resistors in the T-network is linear, solving for the currents is much more difficult. For the network of Fig. 47 the terminal voltages are

$$v_1 = \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} (i_1) + \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} (i_1 + i_2) \qquad (172)$$

$$v_2 = i_2 + \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} (i_1 + i_2) = \begin{pmatrix} 0, & 2 \\ 0, & 3/2 \end{pmatrix} (i_1 + i_2) - i_1 \qquad (173)$$
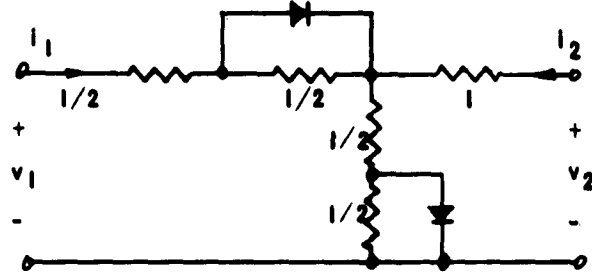
FIG. 47. PWL T-NETWORK WITH TWO PWL RESISTORS.

If we make a three-dimensional plot of v_1 vs i_1 and i_2, the resulting piecewise-planar surface has four planar sections. Examination of Eq. (172) shows that

$$v_1 = 2\,i_1 + i_2 \qquad\qquad (i_1 < 0, \quad i_1 + i_2 < 0)$$

$$v_1 = 3/2\,i_1 + i_2 \qquad\qquad (i_1 > 0, \quad i_1 + i_2 < 0)$$

$$v_1 = 3/2\,i_1 + \tfrac{1}{2}\,i_2 \qquad\qquad (i_1 < 0, \quad i_1 + i_2 > 0)$$

$$v_1 = i_1 + \tfrac{1}{2}\,i_2 \qquad\qquad (i_1 > 0, \quad i_1 + i_2 > 0)$$

Fig. 48a represents a top view of this piecewise-planar surface. The breaklines, $i_1 = 0$ and $i_1 + i_2 = 0$, divide the i_1-i_2 plane into four sections. A similar representation of v_2 has two sections separated by the breakline, $i_1 + i_2 = 0$.

(a)   (b)
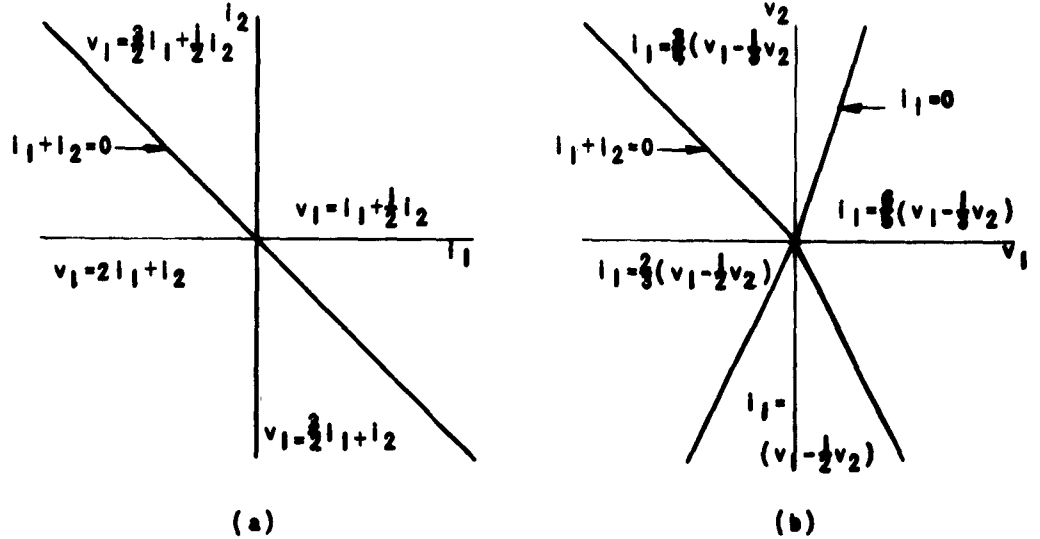
FIG. 48. BREAKLINE PLOTS FOR THE NETWORK OF FIG. 47.

We will now solve Eqs. (172) and (173) to obtain i_1 as a piecewise-planar function of v_1 and v_2. Solving Eq. (173) for ($i_1 + i_2$) and substituting into Eq. (172), we obtain

$$v_1 = \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} (i_1) + \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} \begin{pmatrix} 0, & 1/2 \\ 0, & 2/3 \end{pmatrix} (v_2 + i_1) \qquad (174)$$

Since we do not know how to solve this equation for i_1 directly, we will solve in sections. If $i_1 < 0$,

$$v_1 = i_1 + \begin{pmatrix} 0, & 1/2 \\ 0, & 1/3 \end{pmatrix} (v_2 + i_1) = \begin{pmatrix} 0, & 3/2 \\ 0, & 4/3 \end{pmatrix} (v_2 + i_1) - v_2 \qquad (175)$$

from which

$$i_1 = \begin{pmatrix} 0, & 2/3 \\ 0, & 3/4 \end{pmatrix} (v_1 + v_2) - v_2 \qquad (176)$$

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Similarly, if $i_1 > 0$, we obtain

$$i_1 = \begin{pmatrix} 0, & 1 \\ 0, & 6/5 \end{pmatrix} (v_1 + \tfrac{1}{2} v_2) - v_2 \tag{177}$$

These two equations for i_1 can be combined to yield

$$i_1 = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} \left[\begin{pmatrix} 0, & 2/3 \\ 0, & 3/4 \end{pmatrix} (v_1 + v_2) - v_2 \right]$$

$$+ \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \left[\begin{pmatrix} 0, & 1 \\ 0, & 6/5 \end{pmatrix} (v_1 + \tfrac{1}{2} v_2) - v_2 \right] \tag{178}$$

Given values of v_1 and v_2, Eq. (178) is useful for computing i_1, but the equation does not reveal the nature of the piecewise-planar surface that i_1 represents. Determination of the breaklines of this piecewise-planar surface will greatly aid in its visualization. The breaklines in the v_1-v_2 plane can be related to the breaklines in the i_1-i_2 plane. To find the breakline that corresponds to $i_1 = 0$, we set i_1 equal to zero in Eqs. (172) and (173) and solve for the relationship between v_2 and v_1. With $i_1 = 0$, Eqs. (172) and (173) reduce to

$$v_1 = \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} (i_2) \qquad \text{and} \qquad v_2 = \begin{pmatrix} 0, & 2 \\ 0, & 3/2 \end{pmatrix} (i_2)$$

Eliminating i_2, we obtain

$$v_2 = \begin{pmatrix} 0, & 2 \\ 0, & 3/2 \end{pmatrix} \begin{pmatrix} 0, & 1 \\ 0, & 2 \end{pmatrix} (v_1) = \begin{pmatrix} 0, & 2 \\ 0, & 3 \end{pmatrix} (v_1) \tag{179}$$

Similarly, for the other breakline, substituting $i_1 + i_2 = 0$ into Eqs. (172) and (173) yields

$$v_1 = \begin{pmatrix} 0, & 1 \\ 0, & 1/2 \end{pmatrix} (i_1)$$

$$v_2 = i_2 = -i_1$$

from which

$$v_2 = \begin{pmatrix} 0, & -1 \\ 0, & -2 \end{pmatrix} (v_1) \tag{180}$$

Eqs. (179) and (180) represent breaklines which divide the v_1-v_2 plane into four regions as shown in Fig. 48b.

We next consider an example of a PWL T-network in which R_1, R_2, and R_3 are all PWL. If

$$R_1 = \begin{pmatrix} 0, & 1 \\ 1, & 1/2 \end{pmatrix}_2 \qquad R_2 = \begin{pmatrix} 0, & 1/2 \\ -2, & 1 \end{pmatrix}_4 \qquad R_3 = \begin{pmatrix} -3, & 2 \\ 0, & 1 \end{pmatrix}_3$$

Eqs. (165) and (166) become

$$v_1 = \begin{pmatrix} 0, & 1 \\ 1, & 1/2 \end{pmatrix} (i_1) + \begin{pmatrix} -3, & 2 \\ 0, & 1 \end{pmatrix} (i_1 + i_2) \tag{181}$$

$$v_2 = \begin{pmatrix} 0, & 1/2 \\ -2, & 1 \end{pmatrix} (i_2) + \begin{pmatrix} -3, & 2 \\ 0, & 1 \end{pmatrix} (i_1 + i_2) \tag{182}$$

The breaklines for v_1 are $i_1 = 2$ and $i_1 + i_2 = 3$, and

the breaklines for v_2 are $i_2 = 4$ and $i_1 + i_2 = 3$. These breaklines di-

vide the i_1-i_2 plane into seven regions as shown in Fig. 49a. Even though we cannot solve Eqs. (181) and (182) directly for i_1 and i_2, we can use these equations to solve for the breaklines of i_1 and i_2 in the v_1-v_2 plane. If we substitute $i_1 = 2$ in Eqs. (181) and (182) and then apply the shifting rule, we obtain

$$v_1 = 2 + \begin{pmatrix} -3, & 2 \\ 0, & 1 \end{pmatrix} (2 + i_2) = \begin{pmatrix} 3, & 2 \\ 4, & 1 \end{pmatrix} (i_2) \tag{183}$$

$$v_2 = \begin{pmatrix} 0, & 1/2 \\ -2, & 1 \end{pmatrix} (i_2) + \begin{pmatrix} 1, & 2 \\ 2, & 1 \end{pmatrix} (i_2) = \begin{pmatrix} 1, & 5/2 \\ 2, & 3/2 \\ 0, & 2 \end{pmatrix} (i_2) \tag{184}$$
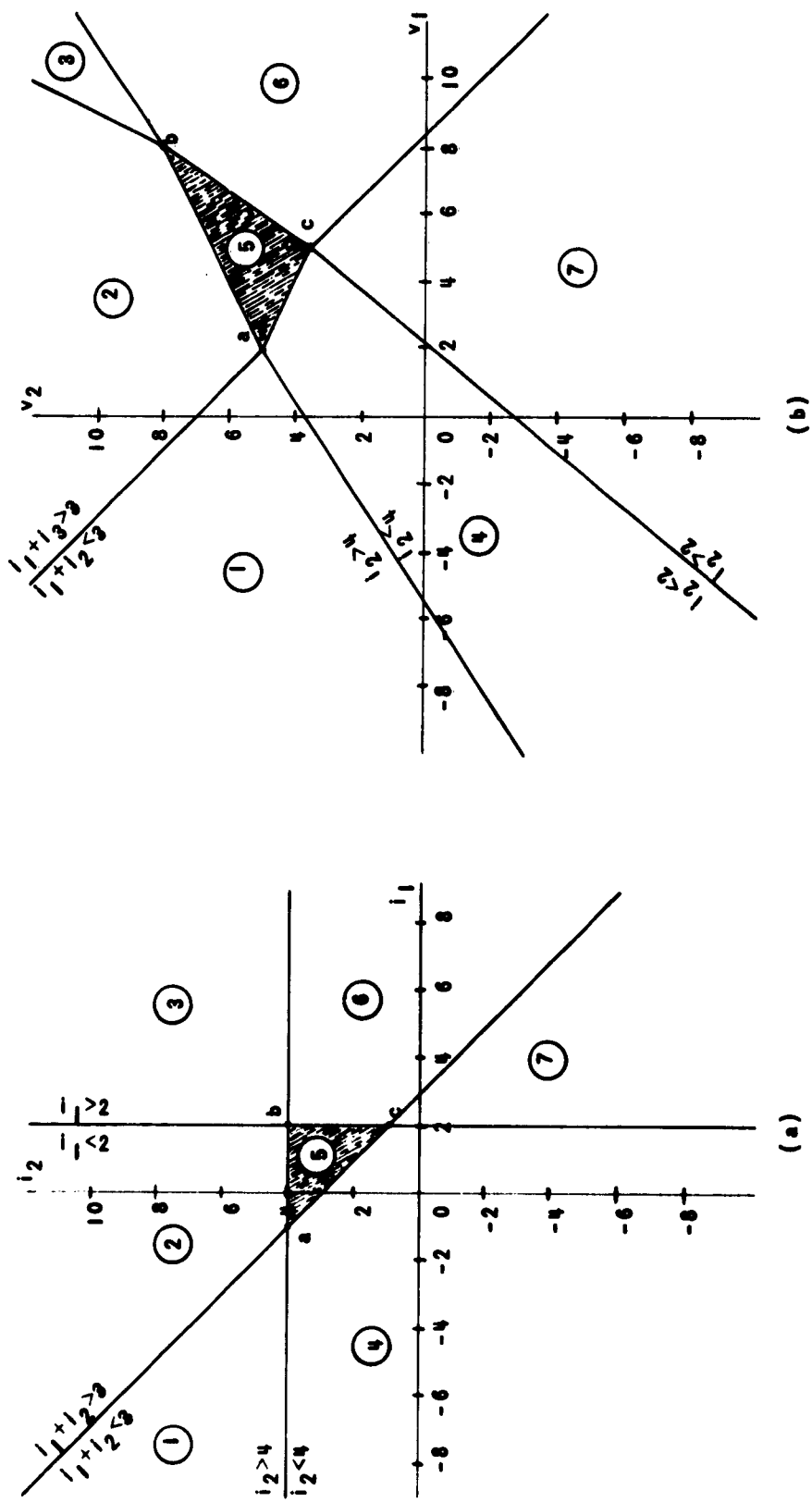
FIG. 49. BREAKLINE PLOTS FOR A PWL T-NETWORK.

Solving Eq. (183) for i_2 and substituting in Eq. (184), we obtain

$$v_2 = \begin{pmatrix} 1, & 5/2 \\ 2, & 3/2 \\ 0, & 2 \end{pmatrix} \begin{pmatrix} -3/2, & 1/2 \\ -4, & 1 \end{pmatrix} (v_1) = \begin{pmatrix} -11/4, & 5/4 \\ -4, & 3/2 \\ -8, & 2 \end{pmatrix} (v_1)$$

This equation represents the breakline in the v_1-v_2 plane that corresponds to the breakline $i_1 = 2$. Similarly, the breaklines that correspond to $i_1 + i_2 = 3$ and $i_2 = 4$ are respectively

$$v_2 = \begin{pmatrix} 7, & -1 \\ 6, & -1/2 \\ 17/2, & -1 \end{pmatrix} (v_1) \qquad \text{and} \qquad v_2 = \begin{pmatrix} 11/3, & 2/3 \\ 4, & 1/2 \\ 8/3, & 2/3 \end{pmatrix} (v_1)$$

These three breaklines are plotted in Fig. 49b. The seven regions in the v_1-v_2 plane are numbered to correspond to the respective regions in the i_1-i_2 plane. One can think of Eqs. (181) and (182) as mapping the breaklines in the i_1-i_2 plane into the corresponding breaklines in the v_1-v_2 plane. Figure 49b also represents a top view of a plot of i_1 or i_2 as a function of v_1 and v_2. The equations for each of the seven regions can be solved individually if desired. For example, in region one, $i_1 < 2$, $i_2 > 4$, and $i_1 + i_2 < 3$, so Eqs. (181) and (182) reduce to

$$v_1 = i_1 - 3 + 2 (i_1 + i_2)$$

$$v_2 = -2 + i_2 - 3 + 2 (i_1 + i_2)$$

from which

$$i_1 = 0.6 \, v_1 - 0.4 \, v_2 - 0.2$$

$$i_2 = -0.4 \, v_1 + 0.6 \, v_2 + 1.8$$

These equations are valid for v_2 in the range

$$11/3 + 2/3 \, v_1 < v_2 < 7 - v_1$$

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Similar calculations can be carried out for the other six regions. Although we were unable to determine the solutions for i_1 and i_2 directly in terms of PWL operators, PWL operators were still useful for determining the breaklines of these solutions.

C. EXPANSION OF PIECEWISE-PLANAR FUNCTIONS IN TERMS OF PWL OPERATORS

When we tried to solve Eqs. (181) and (182) simultaneously for i_1, we were unable to find a solution directly in terms of PWL operators; therefore, it was necessary to determine the breaklines first and then solve for each section separately. We know that i_1 is a piecewise-planar function of v_1 and v_2, but it is not obvious that this function can be expressed in terms of PWL operators. In general, we would like to determine what class of piecewise-planar functions of two variables can be expressed in terms of PWL operators.

Stern [Ref. 5, pp. 59-63] gives a method for expressing any single-valued piecewise-planar function of two variables in terms of ϕ^+ - transformations. The piecewise-planar surface is first broken down into a sum of pyramids, each with a vertex on one of the breakpoints, and then each pyramid is expressed in terms of ϕ^--transformations. We have shown in Section II. E. that any PWL function which can be expressed in Stern's notation can be converted to PWL-operator notation. It therefore follows that any single-valued piecewise-planar function of the two variables can be expressed in terms of PWL operators. The method for finding such an expression is somewhat devious and the resulting expression is somewhat cumbersome to work with, but at least such an expression always exists.

As an example, consider the piecewise-planar surface whose breaklines are shown in Fig. 48b. Since this surface has a single breakpoint, it can be written directly in Stern's notation as

$$i_1 = \left[2/3 \left(v_1 - \tfrac{1}{2} v_2\right), \ 3/4 \left(v_1 - 1/3 v_2\right), \ \left(v_1 - \tfrac{1}{2} v_2\right), \ 6/5 \left(v_1 - 1/3 v_2\right) \right] \phi^+$$

$$(185)$$

Using Eqs. (22) and (A-5) to convert to PWL operator notation, we obtain

$$
\begin{aligned}
i_1 &= \left[\begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \left(\tfrac{1}{12} v_1 + \tfrac{1}{12} v_2 \right) + \left(\tfrac{2}{3} v_1 - \tfrac{1}{3} v_2 \right) \right. \\
&\quad \left. + \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \left(\tfrac{1}{5} v_1 + \tfrac{1}{10} v_2 \right) + \left(v_1 - \tfrac{1}{2} v_2 \right) \right] \phi^+ \\
&= \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \left[\begin{pmatrix} 0, & 0 \\ 0, & 6/5 \end{pmatrix} \left(v_1 + \tfrac{1}{2} v_2 \right) - \begin{pmatrix} 0, & 0 \\ 0, & 3/4 \end{pmatrix} \left(v_1 + v_2 \right) \right] \\
&\quad + \begin{pmatrix} 0, & 0 \\ 0, & 3/4 \end{pmatrix} \left(v_1 + v_2 \right) - v_2
\end{aligned}
\tag{186}
$$

This expression is more complicated than Eq. (178) which was obtained more directly by solving Eqs. (172) and (173).

D. PIECEWISE-PLANAR OPERATORS

In the previous section, we showed that any single-valued piecewise-planar function can be expressed in terms of PWL operators. As seen from the example, for even a simple piecewise-planar surface, the resulting expression may be very complicated, and the expression which represents a given piecewise-planar function is not unique. Furthermore, there is no simple and direct method for writing down a PWL-operator expansion of the function from its graph.

For the above reasons, it would be desirable to define piecewise-planar operators to represent piecewise-planar functions more directly. If a suitable piecewise-planar operator, $\tilde{q}$, could be defined, we could write the characteristics of a PWL two-port in the form

$$
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \tilde{q} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}
\tag{187}
$$

Algebraic operations with piecewise-planar operators could then be de-

fined by analogy with PWL operators. For example, in terms of the inversion operation, Eq. (187) could be solved for the currents in terms of the voltages in the form

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = Q^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{188}$$

Several attempts have been made to define piecewise-planar operators, but no really satisfactory method was discovered except in the special cases where all of the breaklines are parallel or all of the breaklines meet at a single point. A basic difficulty is encountered in trying to extend the PWL-operator concept to the piecewise-planar case. A linear segment of a PWL curve can be adjacent to only two other segments, but a planar section of a piecewise-planar surface can be adjacent to any number of other sections. This limitation makes a simple extension of the PWL operator notation impossible. Determination of a satisfactory method for generalizing PWL operators to the case of two or more independent variables is one of the important unsolved problems in PWL-network theory.

VII. ANALYSIS OF ELECTRONIC CIRCUITS

PWL operators are useful in the analysis of electronic circuits. For large-signal operation, the characteristics of diodes, vacuum tubes, transistors, and other electronic devices can be approximated by PWL characteristics and described in terms of PWL operators. Since a number of diode-circuit examples have already been presented, in this chapter we will analyze several vacuum-tube and transistor circuits.

A. TRIODE CHARACTERISTICS

Figure 50a shows a PWL approximation to the characteristics of a triode. If a three-dimensional plot of i_b as a function of e_b and e_g is made, the breaklines divide the e_g-e_b plane into three regions (Fig. 50b). In the cutoff region, $i_b = 0$; in the normal-operation region, $i_b = (e_b + \mu e_g)/r_p$; and in the saturation region, $i_b = e_b/r_s$. A PWL triode model [Ref. 2, p. 227] that has these characteristics is shown in Fig. 50c. When D_1 is off, $i_b = 0$ and the tube is cut off. When D_1 is on and D_2 is off, the tube is in the normal operating region. When D_1 and D_2 are both on, $i_b = e_b/r_s$ and the tube is saturated. D_3 and r_g have been added to the model to account for the grid current which flows when $e_g > 0$.

We will now derive the PWL-operator equations that represent the characteristics of the PWL triode model. We can redraw the PWL model as shown in Fig. 50d by transforming the dependent voltage source to a current source. From the equivalent model,

$$
e_b = \begin{pmatrix} 0, & 0 \\ 0, & r_p\text{-}r_s \end{pmatrix} \begin{pmatrix} i_b - \dfrac{\mu e_g}{r_p\text{-}r_s} \end{pmatrix} + \begin{pmatrix} 0, & \infty \\ 0, & r_s \end{pmatrix} (i_b) \tag{189}
$$

We will solve this equation to obtain i_b as a function of e_b and e_g. If $i_b > 0$, Eq. (189) reduces to

$$
e_b = \begin{pmatrix} 0, & 0 \\ 0, & r_p\text{-}r_s \end{pmatrix} \begin{pmatrix} i_b - \dfrac{\mu e_g}{r_p\text{-}r_s} \end{pmatrix} + r_s\, i_b \tag{189a}
$$

(a) PWL TRIODE CHARACTERISTICS.

(b) BREAKLINES IN e_g-e_b PLANE.

(c) PWL MODEL OF TRIODE.

(d) EQUIVALENT PWL MODEL.
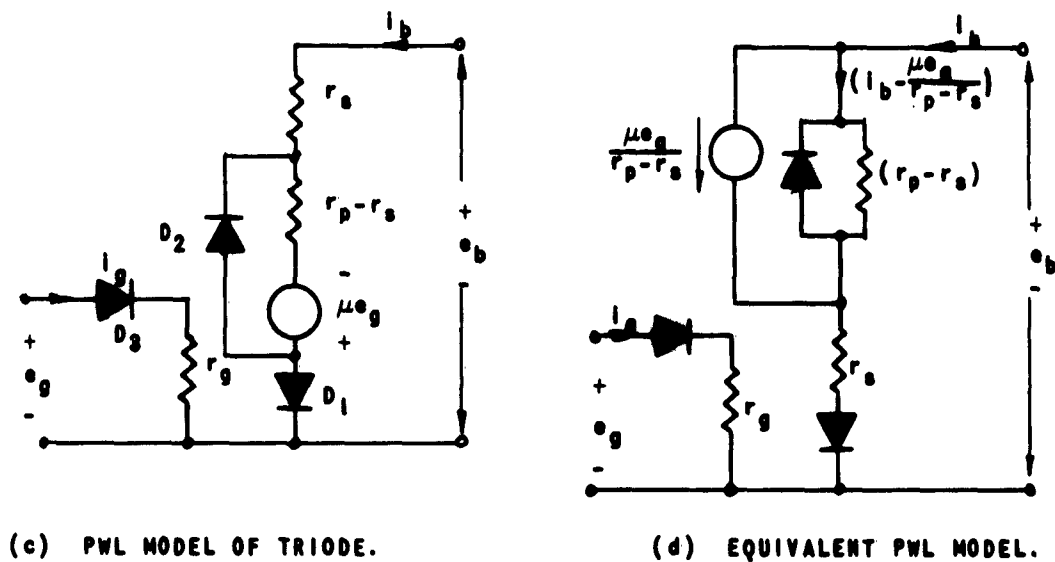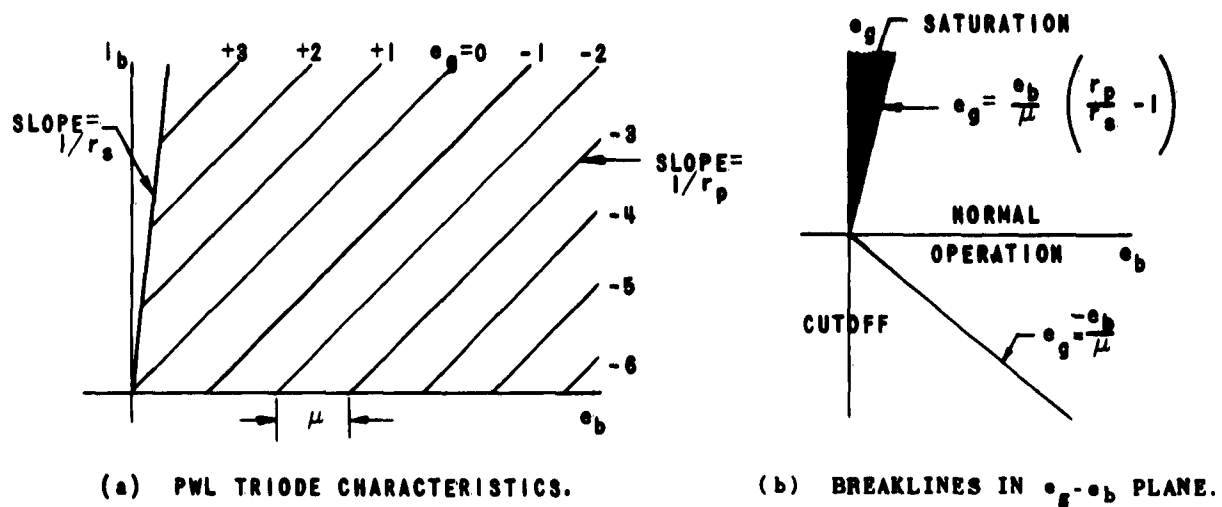
FIG. 50. PWL TRIODE.

Multiplying both sides of this equation by $1/r_s$ and then subtracting $\mu e_g/(r_p - r_s)$, we obtain

$$\left(\frac{e_b}{r_s} - \frac{\mu e_g}{r_p - r_s}\right) = \begin{pmatrix} 0, & 0 \\ 0, & \frac{r_p}{r_s} - 1 \end{pmatrix} \left(i_b - \frac{\mu e_g}{r_p - r_s}\right) + \left(i_b - \frac{\mu e_g}{r_p - r_s}\right)$$

$$= \begin{pmatrix} 0, & 1 \\ 0, & r_p/r_s \end{pmatrix} \left(i_b - \frac{\mu e_g}{r_p - r_s}\right) \qquad (190)$$

Solving for i_b and then subtracting e_b/r_s from both sides, we obtain

$$i_b - \frac{e_b}{r_s} = \begin{pmatrix} 0, & 1 \\ 0, & r_s/r_p \end{pmatrix} \left(\frac{e_b}{r_s} - \frac{\mu e_g}{r_p - r_s}\right) + \left(\frac{\mu e_g}{r_p - r_s} - \frac{e_b}{r_s}\right)$$

$$= \begin{pmatrix} 0, & 0 \\ 0, & \frac{r_s}{r_p} - 1 \end{pmatrix} \left(\frac{e_b}{r_s} - \frac{\mu e_g}{r_p - r_s}\right) \qquad (191)$$

This equation can be rewritten in the form

$$i_b = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} \left[\frac{1}{r_p}(e_b + \mu e_g) - \frac{e_b}{r_s}\right] + \frac{e_b}{r_s} \qquad (192)$$

When this expression is positive, it gives the correct value of i_b. To prevent i_b from going negative, we premultiply by $\begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix}$.

The resulting expression for the plate current in the PWL triode model is

$$i_b = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \left\{ \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} \left[\frac{1}{r_p}(e_b + \mu e_g) - \frac{e_b}{r_s}\right] + \frac{e_b}{r_s} \right\} \qquad (193)$$

and the grid current is

$$i_g = \begin{pmatrix} 0, & 0 \\ 0, & 1/r_g \end{pmatrix} (e_g) \tag{194}$$

B. TRANSISTOR CHARACTERISTICS

Using the PWL model of Fig. 51, the characteristics of a transistor in the grounded-base connection can be represented by the equations

$$v_{eb} = \begin{pmatrix} 0, & \infty \\ 0, & r_e \end{pmatrix} (i_e) + r_b (i_e + i_c) = \begin{pmatrix} 0, & \infty \\ 0, & r_e+r_b \end{pmatrix} (i_e) + r_b i_c \tag{195}$$

and

$$v_{cb} = \begin{pmatrix} 0, & r_c \\ 0, & 0 \end{pmatrix} (i_c + ai_e) + r_b (i_e + i_c) \tag{196}$$

$$= \begin{pmatrix} 0, & r_c+r_b \\ 0, & r_b \end{pmatrix} (i_c + ai_e) + r_b(1-a) i_e \tag{196}$$

Replacing i_e with $-(i_b + i_c)$ in the above equations, we obtain the equations which describe the grounded-emitter transistor model of Fig. 52a as

$$v_{be} = -v_{eb} = \begin{pmatrix} 0, & r_e \\ 0, & \infty \end{pmatrix} (i_b + i_c) + r_b i_b \tag{197}$$

$$v_{ce} = v_{cb} + v_{be} = \begin{pmatrix} 0, & r_e \\ 0, & \infty \end{pmatrix} (i_b + i_c) + \begin{pmatrix} 0, & r_c \\ 0, & 0 \end{pmatrix} [i_c - a (i_b + i_c)] \tag{198}$$

Equation (198) can be rewritten in the form

$$v_{ce} = \begin{pmatrix} 0, & r_e \\ 0, & \infty \end{pmatrix} (i_b + i_c) + \begin{pmatrix} 0, & r_d \\ 0, & 0 \end{pmatrix} (i_c - bi_b) \tag{199}$$
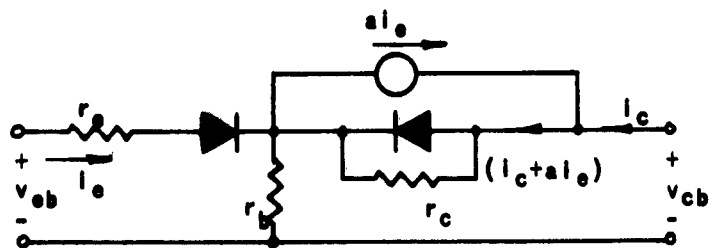
FIG. 51.   PWL MODEL FOR GROUNDED-BASE TRANSISTOR.

where $r_d = r_c (1 - a)$ and $b = a/(1 - a)$. Equations (197) and (199) correspond to the alternate grounded-emmiter model of Fig. 52b.

(a)

(b)
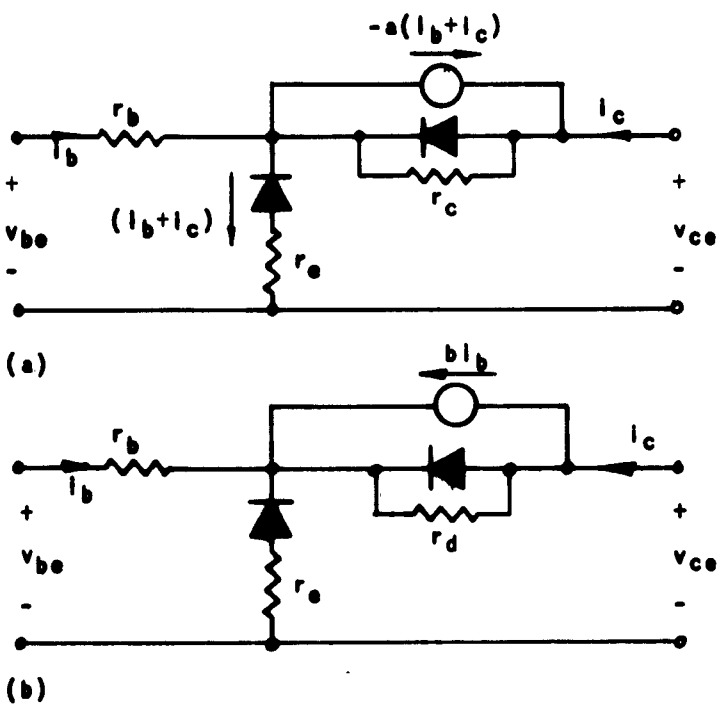
FIG. 52.   PWL MODELS FOR GROUNDED-EMITTER TRANSISTOR.

C. TRANSFER CHARACTERISTIC OF A TRIODE FEEDBACK AMPLIFIER

As an example of the analysis of a vacuum-tube circuit, we will determine the voltage transfer characteristic of the triode feedback amplifier of Fig. 53.[*] We will use the PWL model of the triode (Fig. 50) with the following values of tube parameters:

$$r_p = 5 \text{ k}\Omega, \quad \mu = 20, \quad r_s = 0.1 \text{ k}\Omega, \quad \text{and} \quad r_g = 500 \text{ k}\Omega$$

Substituting these values into Eqs. (193) and (194), we find that the tube characteristics are given by

$$i_b = \begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \left[\begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} (4e_g - 9.8 \, e_b) + 10 \, e_b \right] \tag{200}$$

$$i_g = \begin{pmatrix} 0, & 0 \\ 0, & .002 \end{pmatrix} (e_g) \tag{201}$$

Since the 1000-kΩ feedback resistor is a negligible load on the plate circuit, we can write the following node equations for the network:

$$i_b = \frac{200 - e_b}{5} \tag{202}$$

and

$$i_g = \frac{e_b}{1000} + \frac{e_i}{1000} - \frac{3e_g}{1000} \tag{203}$$

Combining Eqs. (200) and (202) and Eqs. (201) and (203) and simplifying, we obtain

$$(200 - e_b) = \begin{pmatrix} 0, & 0 \\ 0, & 5 \end{pmatrix} \left[\begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} (4e_g - 9.8 \, e_b) + 10 \, e_b \right] \tag{204}$$

---

[*] Stern [Ref. 5, pp. 23-25] analyzes this same network using ϕ^+-transformations. The amount of work required by the two methods is about the same.

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and

$$e_i = 3e_g + \begin{pmatrix} 0, & 0 \\ 0, & 2 \end{pmatrix} (e_g) - e_b = \begin{pmatrix} 0, & 3 \\ 0, & 5 \end{pmatrix} (e_g) - e_b$$

$$(205)$$
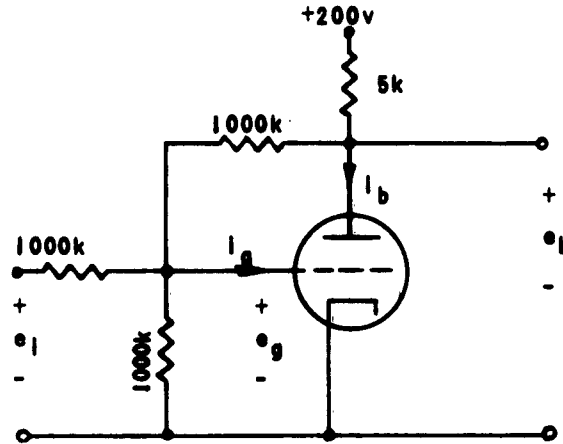
FIG. 53. TRIODE FEEDBACK AMPLIFIER.

Equation (204) can be solved for e_g as follows. Premultiplying by

$\begin{pmatrix} 0, & 0 \\ 0, & 5 \end{pmatrix}^{-1}$, applying the shifting rule, (Eq. (59), and subtracting

$10e_b$ yields

$$\begin{pmatrix} 0, & L \\ 0, & .2 \end{pmatrix} (200 - e_b) - 10e_b = \begin{pmatrix} 200L, & L \\ 40, & 10.2 \end{pmatrix} (-e_b) = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix} (4e_g - 9.8\,e_b)$$

Premultiplying by $\begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix}^{-1}$ and adding $9.8\ e_b$ gives

$$\begin{pmatrix} 0, & 1 \\ 0, & L \end{pmatrix} \begin{pmatrix} 200\ L, & L \\ 40, & 10.2 \end{pmatrix} (-e_b) + 9.8\ e_b = \begin{pmatrix} 200L, & L \\ 40, & 0.4 \\ 40L, & 10.2L \end{pmatrix} (-e_b) = 4e_g$$

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Solving for e_g and substituting into Eq. (205) we obtain

$$e_i = \begin{pmatrix} 0, & 3 \\ 0, & 5 \end{pmatrix} \begin{pmatrix} 50L, & 0.25L \\ 10, & 0.1 \\ 10L, & 2.55L \end{pmatrix} (-e_b) - e_b = \begin{pmatrix} 150L, & 0.75L \\ 30, & 1.3 \\ 50, & 1.5 \\ 50L, & 12.75L \end{pmatrix} (-e_b)$$

Solving for e_b and letting $L \to \infty$, the desired transfer characteristic is

$$e_b = \begin{pmatrix} 200, & 0 \\ 300/13, & -10/13 \\ 100/3, & -2/3 \\ 200/51, & 0 \end{pmatrix} (e_i) \qquad (206)$$

D. TRANSFER CHARACTERISTIC OF A TRANSISTOR AMPLIFIER

As an example of analysis of a transistor circuit, we will derive the voltage transfer characteristic of the grounded-base transistor amplifier of Fig. 54a. Using the PWL model of Fig. 54b, the amplifier can be described by the equations

$$v_1 = \begin{pmatrix} 0, & L \\ 0, & R_e + r_b \end{pmatrix} (i_e) + r_b \, i_c \qquad (207)$$

$$v_2 = \begin{pmatrix} E, & L \\ E, & 0 \end{pmatrix} (i_c + a i_e) + r_b (i_e + i_c) = -i_c R_c \qquad (207a)$$

Equation (207a) can be rewritten in the form

$$\begin{pmatrix} E, & L \\ E, & r_b + R_c \end{pmatrix} (i_c + a i_e) = [(a - 1) r_b + a R_c] i_e$$

Solving for i_c, the current transfer characteristic is

$$i_c = \begin{pmatrix} \dfrac{-E}{L}, & -a \\ \dfrac{-E}{r_b + R_c}, & \dfrac{-r_b}{r_b + R_c} \end{pmatrix} (i_e)$$

(a)

(b)

(c)
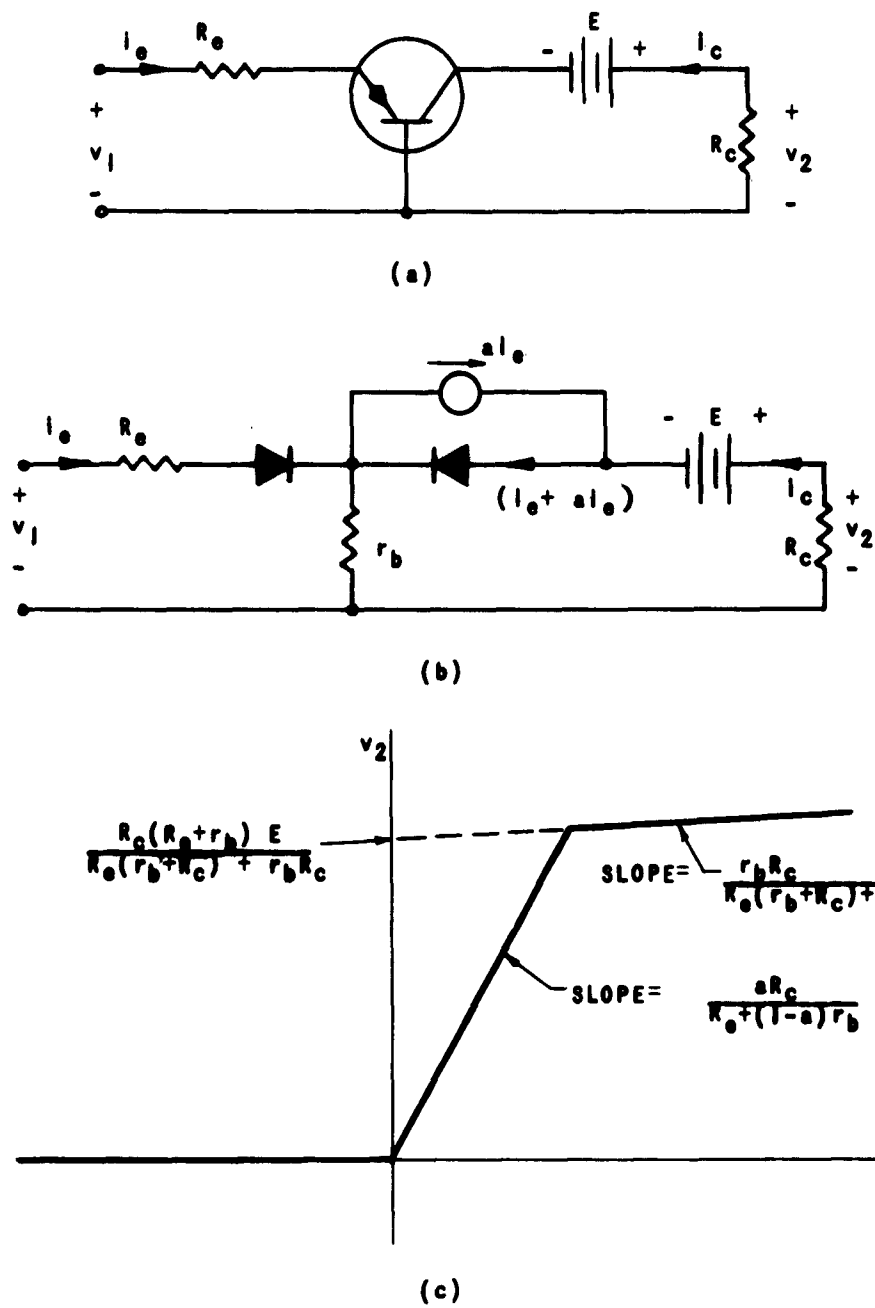
FIG. 54. GROUNDED-BASE TRANSISTOR AMPLIFIER AND TRANSFER CURVE.

from which

$$v_2 = -R_c \, i_c = \begin{pmatrix} \dfrac{ER_c}{L}, & aR_c \\[3mm] \dfrac{ER_c}{r_b+R_c}, & \dfrac{r_b R_c}{r_b+R_c} \end{pmatrix} (i_e)$$

Solving this equation for i_e and substituting in Eq. (207), we obtain

$$v_1 = \begin{pmatrix} 0, & L \\[2mm] 0, & R_e+r_b \end{pmatrix} \begin{pmatrix} \dfrac{-E}{aL}, & \dfrac{1}{aR_c} \\[3mm] \dfrac{-E}{r_b}, & \dfrac{r_b+R_c}{r_b R_c} \end{pmatrix} (v_2) + r_b \begin{pmatrix} \dfrac{-v_2}{R_c} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{-E}{a}, & \dfrac{L}{aR_c} \\[4mm] \dfrac{-(R_e+r_b)E}{aL}, & \dfrac{R_e+r_b}{aR_c} - \dfrac{r_b}{R_c} \\[4mm] \dfrac{-(R_e+r_b)E}{r_b}, & \dfrac{(R_e+r_b)(R_c+r_b)}{r_b R_c} - \dfrac{r_b}{R_c} \end{pmatrix} (v_2)$$

Solving for v_2 and letting $L \to \infty$, we obtain the voltage characteristic (Fig. 54c) as

$$v_2 = \begin{pmatrix} 0, & 0 \\[3mm] 0, & \dfrac{aR_c}{R_e+(1-a)r_b} \\[4mm] \dfrac{R_c(R_e+r_b)\,E}{R_e(r_b+R_c)+r_b R_c}, & \dfrac{r_b R_c}{R_e(r_b+R_c)+r_b R_c} \end{pmatrix} (v_1) \qquad (208)$$

E. SERIES-TRIODE NEGATIVE RESISTANCE CIRCUIT

We will now determine the input v-i characteristic of the two-triode circuit shown in Fig. 55a. Each triode is replaced with a PWL model as shown in Fig. 55b. The grid voltages are

$$e_{g1} = -R_k i_2 \qquad e_{g2} = -R_k(i_2 - i_1)$$

Since the sum of the voltages around the loop is zero,

$$E = \left[r_p + R_k + \mu R_k + \begin{pmatrix} 0, & L \\ 0, & 0 \end{pmatrix} \right] (i_2 - i_1)$$

$$+ \left[r_p + R_k + \mu R_k + \begin{pmatrix} 0, & L \\ 0, & 0 \end{pmatrix} \right] (i_2) \qquad (209)$$

The input voltage is

$$v = \left[r_p + R_k + \begin{pmatrix} 0, & L \\ 0, & 0 \end{pmatrix} \right] (i_2) + \mu R_k (i_2 - i_1) \qquad (210)$$

If we let $R = r_p + R_k + \mu R_k$ and then rearrange the terms, these equations become

$$E = \begin{pmatrix} 0, & L \\ 0, & R \end{pmatrix} (i_2 - i_1) + \begin{pmatrix} 0, & L \\ 0, & R \end{pmatrix} (i_2) \qquad (211)$$

and

$$v = \begin{pmatrix} 0, & L \\ 0, & R \end{pmatrix} (i_2) - \mu R_k i_1 \qquad (212)$$

From Eq. (211),

$$\begin{pmatrix} 0, & R \\ 0, & L \end{pmatrix} (i_1 - i_2) = \begin{pmatrix} -E, & L \\ -E, & R \end{pmatrix} (i_2)$$

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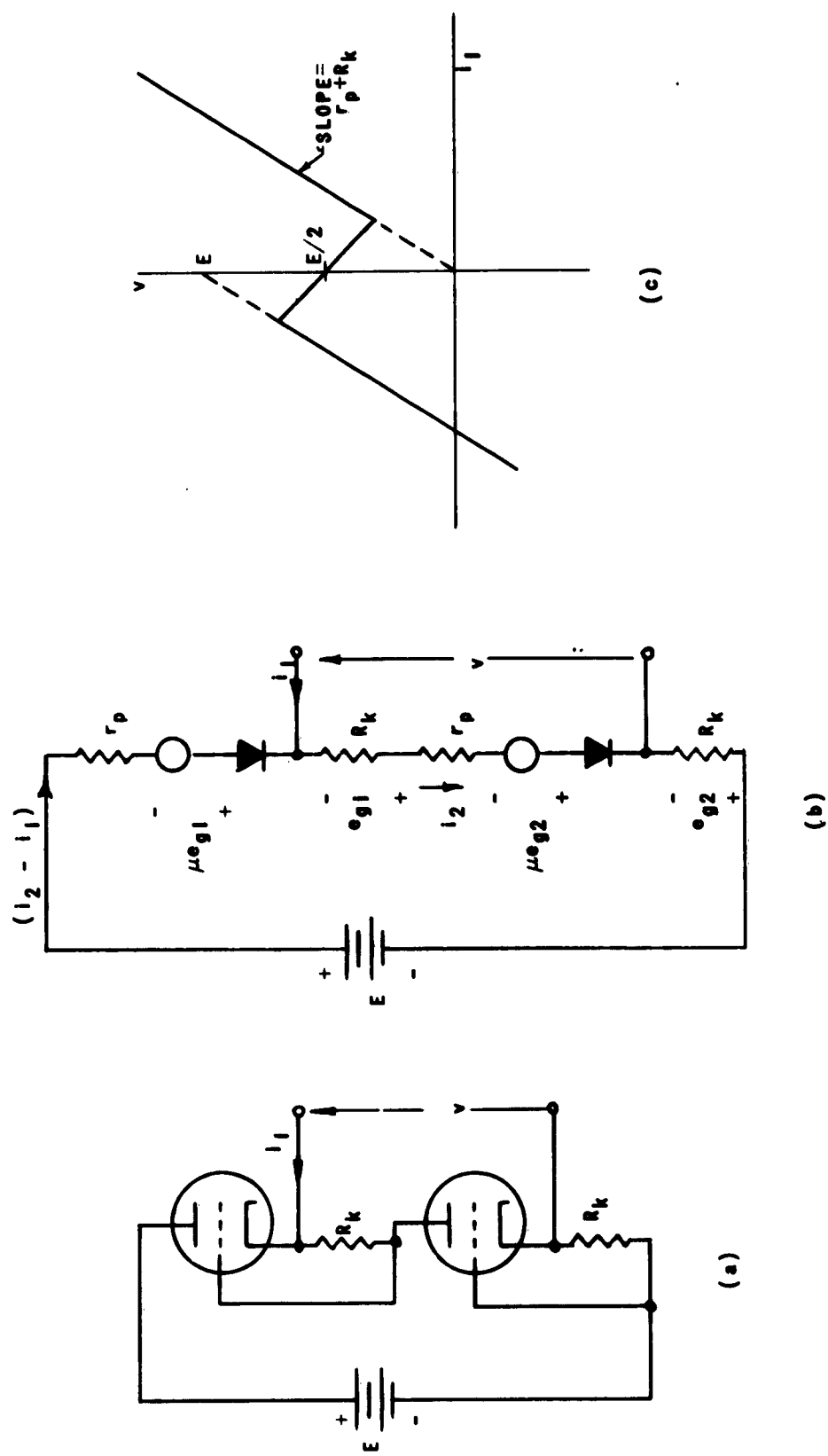
FIG. 55.   SERIES-TRIODE NEGATIVE RESISTANCE CIRCUIT.

Solving for i_1, we obtain

$$i_1 = \begin{pmatrix} 0, & 1/R \\ 0, & 1/L \end{pmatrix} \begin{pmatrix} -E, & L \\ -E, & R \end{pmatrix} (i_2) + i_2 = \begin{pmatrix} -E/R, & L/R \\ -E/R, & 2 \\ -E/L, & 1 \end{pmatrix} (i_2)$$

We next solve for i_2 and substitute in Eq. (212), which yields

$$v = \begin{pmatrix} 0, & L \\ & \\ 0, & R \end{pmatrix} \begin{pmatrix} E/L, & R/L \\ E/2R, & 1/2 \\ E/L, & 1 \end{pmatrix} (i_1) - \mu R_k \, i_1 = \begin{pmatrix} E, & R \\ E/2, & R/2 \\ 0, & R \end{pmatrix} (i_1) - \mu R_k \, i_1$$

Replacing R with $r_p + R_k + \mu R_k$ in the above equation, we find the input v-i characteristic to be

$$v = \begin{pmatrix} E, & r_p + R_k \\ \dfrac{E}{2}, & \dfrac{r_p + R_k \,(1-\mu)}{2} \\ 0, & r_p + R_k \end{pmatrix} (i) \qquad (213)$$

This curve is plotted in Fig. 55c for the case $R_k \,(\mu-1) > r_p$.

VIII. ENERGY STORAGE ELEMENTS IN PWL NETWORKS

Until now, we have considered only resistive PWL networks without energy-storage elements. In this chapter, we will attempt to extend the usefulness of PWL operators to the analysis of networks that also contain linear or PWL capacitors or inductors. By a PWL capacitor, we mean a capacitor whose charge-voltage characteristic is PWL. If we represent this characteristic by the PWL operator C, then the current through the capacitor is

$$i = \frac{dq}{dt} = \frac{d}{dt} [C(v)]$$

By a PWL inductor, we mean an inductor whose flux-linkage - current characteristic is PWL. If we represent this characteristic by L, then the voltage across the inductor is

$$v = \frac{d\phi}{dt} = \frac{d}{dt} [L(i)]$$

PWL inductors and capacitors are useful as approximations to nonlinear inductors and capacitors.

A PWL R-L-C network can always be solved on a section-by-section basis. At any instant of time, the network reduces to a linear network. The solution to the linear differential equations of this network is valid until one of the PWL elements changes state. When the network contains several PWL elements, we must solve for the voltage or current in each one in order to find out which element will change state first. After the change of state, we have a new linear-network problem and a new set of linear differential equations to solve. The constants in the solution can be obtained by matching boundary conditions at the time of transition from one section to the next. This matching process will generally require the numerical solution of transcendental equations. This section-by-section method is usually very tedious and time-consuming to carry out.

An attempt has been made to develop more efficient methods of solving PWL networks that contain energy-storage elements. By using PWL operators, a problem can be formulated in terms of PWL differential equations. For

PWL R-C, R-L, and L-C networks, the solution to these PWL differential equations can be expressed in terms of PWL operators. An efficient method of obtaining the solution has been derived for the PWL parallel R-C network.

A. PARALLEL R-C PWL NETWORKS

In this section, we will develop methods for analyzing the parallel R-C PWL network of Fig. 56. Either the resistor or capacitor, or both, may be PWL. By duality, the methods developed here can be applied to the series R-L network.

The voltage across the parallel R-C network satisfies the differential equation

$$\frac{d}{dt} [C(v)] + R^{-1} (v) = 0 \tag{214}$$

where $C(v)$ is the charge on the capacitor and $R^{-1} (v)$ is the current through the resistor. We will assume that when $t = 0$, the capacitor is initially charged to a positive voltage v_o. If the capacitor is linear, Eq. (214) reduces to

$$c \frac{dv}{dt} + R^{-1} (v) = 0 \tag{215}$$

We will show that if R is monotonic, Eqs. (214) and (215) have solutions of the form

$$v = R[e^{B(-t)}] \tag{216}$$

Taken by itself, e^B does not have any meaning; therefore, to evaluate Eq. (216) for a given value of t, we must first find $a = B(-t)$, then calculate $b = e^a$, and finally find $v = R(b)$. The current in the resistor is given by

$$i = R^{-1} (v) = e^{B(-t)} \tag{217}$$
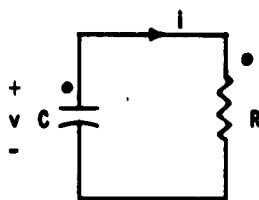
FIG. 56. PARALLEL R-C PWL NETWORK.

We will now attempt to determine B so that Eq. (216) satisfies Eq. (214) and the initial condition $v = v_o$. Substituting Eq. (216) into Eq. (214), we obtain

$$\frac{d}{dt} [CR(e^{B(-t)})] + e^{B(-t)} = 0 \tag{218}$$

The indicated differentiation causes some difficulty because the derivative of a PWL function is not continuous. We will let

$$A = CR = \begin{pmatrix} a_1, \ b_1 \\ a_2, \ b_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n, \ b_n \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \end{matrix} \quad \text{and} \quad B = \begin{matrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_{n-1} \end{matrix} \begin{pmatrix} c_1, \ d_1 \\ c_2, \ d_2 \\ \cdot \\ \cdot \\ c_n, \ d_n \end{pmatrix}$$

be monotonically increasing PWL operators, where the breakpoints of A are designated by $x_1, x_2, \ldots, x_{n-1}$ and the inverse breakpoints of B by $y_1, y_2, \ldots, y_{n-1}$. If some of the x's are negative, enough of the sections of CR should be discarded so that only positive x's remain.[*] Applying the rule for differentiating a function of a function, we obtain

$$\frac{d}{dt} \left\{ A[e^{B(-t)}] \right\} = A'[e^{B(-t)}] \times \frac{d}{dt} [e^{B(-t)}] \tag{219}$$

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[*] This procedure is justified because in Eq. (218) CR operates on $e^{B(-t)}$, which is always positive.

where the prime indicates differentiation with respect to the argument of the function. For the jth section of A,

$$A[e^{B(-t)}] = a_j + b_j e^{B(-t)}$$

from which

$$A'[e^{B(-t)}] = b_j$$

Since the slope of a PWL function is discontinuous, the derivative of A is a jump function of $e^{B(-t)}$, as shown in Fig. 57a. The jumps occur at the breakpoints of A. For the jth section of B,

$$e^{B(-t)} = e^{(c_j - d_j t)}$$

from which

$$\frac{d}{dt}[e^{B(-t)}] = -d_j e^{(c_j - d_j t)} = -d_j e^{B(-t)}$$

Since the value of d_j changes at each breakpoint of B, this derivative is also discontinuous, as shown in Fig. 57b. At the jth breakpoint of B, $e^{B(-t)}$ has the value $e^{(y_j)}$.

Substituting Eq. (219) into Eq. (218), we obtain

$$A'[e^{B(-t)}] \times \frac{d}{dt}[e^{B(-t)}] = -e^{B(-t)} \qquad (220)$$

For the jth section, Eq. (220) becomes

$$b_j \times -d_j \, e^{B(-t)} = -e^{B(-t)}$$

from which

$$d_j = 1/b_j \qquad (j = 1,2,\ldots,n) \qquad (221)$$

Since

$$b_j \neq b_{j+1} \quad \text{and} \quad d_j \neq d_{j+1}, \quad \text{if} \quad b_j d_j = 1, \quad \text{then}$$

$b_j d_{j+1} \neq 1$ and $b_{j+1} d_j \neq 1$. Therefore, in order that Eq. (220) be satisfied for all values of $e^{B(-t)}$, the breakpoints must match as indicated in Fig. 59 , which implies that

$$x_j = e^{y_j} \qquad (j = 1,2,\ldots,n-1)$$

from which

$$y_j = \frac{c_j d_{j+1} - c_{j+1} d_j}{d_{j+1} - d_j} = \ell n \ x_j \qquad (222)$$
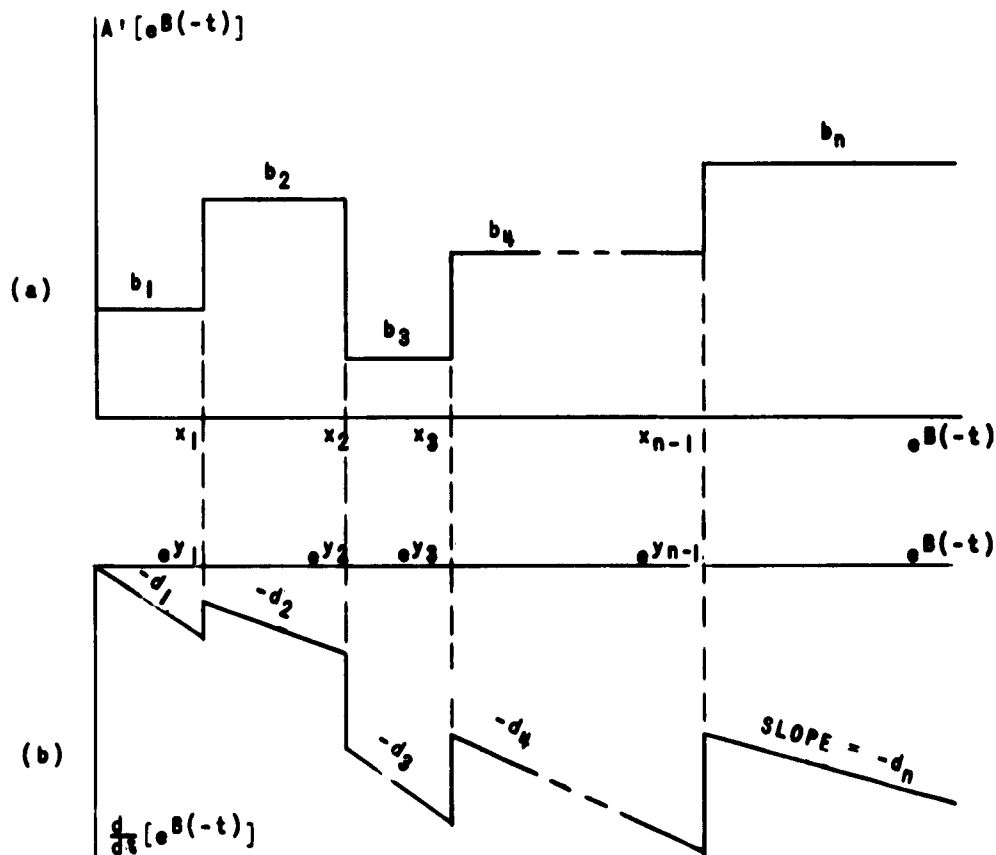
FIG. 57. DERIVATIVES OF PWL FUNCTIONS.

Solving for c_j , we obtain

$$c_j = \frac{(d_{j+1} - d_j) \, \ln x_j + c_{j+1} \, d_j}{d_{j+1}} \qquad (j = 1,2,\ldots,n-1)$$

(223)

Assuming that v_o is chosen so that the initial state of the network lies beyond the last breakpoint,

$$v_o = R[e^{B(0)}] = q_m + r_m e^{c_n}$$

where q_m and r_m are the intercept and slope of the last section of R, from which

$$c_n = \ln \left(\frac{v_o - q_m}{r_m} \right)$$

(224)

We have now determined the unknown PWL operator, B. The d's are given by Eq. (221) and, after Eq. (224) is used to find c_n , Eq. (223) can be used as a recursion formula to find the remaining c's.

We will illustrate the above procedure for analyzing the parallel R-C PWL network by two examples--one with C linear and one with both R and C PWL. If $C = 1$, $v_o = 6$, and

$$R = \begin{pmatrix} -1, & 1 \\ 0, & 1/2 \\ -6, & 2 \\ 3/2, & 1/2 \end{pmatrix} \begin{matrix} 2 \\ 4 \\ 5 \end{matrix} ,$$

then $A = R$ and

$$B = \begin{pmatrix} c_1, & 1 \\ c_2, & 2 \\ c_3, & 1/2 \\ c_4, & 2 \end{pmatrix}$$

The c's are determined from Eq. (224) and (223) as follows:

$$c_4 = \ln \left(\frac{6 - 3/2}{\frac{1}{2}} \right) = 2.1972$$

$$c_3 = \frac{(2-\frac{1}{2}) \ln 5 + \frac{1}{2} c_4}{2} = 1.7564$$

$$c_2 = \frac{(\frac{1}{2}-2) \ln 4 + 2c_3}{\frac{1}{2}} = 2.8667$$

$$c_1 = \frac{(2-1) \ln 2 + c_2}{2} = 1.7799$$

The voltage is then given by

$$v = \begin{pmatrix} -1, & 1 \\ 0, & 1/2 \\ -6, & 2 \\ 3/2, & 1/2 \end{pmatrix} \left[\exp \begin{pmatrix} 1.7799, & 1 \\ 2.8667, & 2 \\ 1.7564, & \frac{1}{2} \\ 2.1972, & 2 \end{pmatrix} (-t) \right] \qquad (225)$$

This solution means

$$v = 3/2 + \tfrac{1}{2} \exp (2.1972 - 2t) = 3/2 + 9/2 \, e^{-2t} \qquad (0 < t < t_1)$$

$$v = -6 + 2 \exp (1.7564 - \tfrac{1}{2}t) = -6 + 10 \exp [-\tfrac{1}{2}(t-t_1)] \quad (t_1 < t < t_2)$$

$$v = \tfrac{1}{2} \exp (2.8667 - 2t) = \tfrac{1}{2} \exp [-2 (t-t_2)] \qquad (t_2 < t < t_3)$$

$$v = -1 + \exp (1.7799 - t) = -1 + 2 \exp [-(t-t_3)] \qquad (t_3 < t)$$

where $t_1 = 0.2939$, $t_2 = 0.7402$, and $t_3 = 1.0868$

These results can be checked by solving the problem on a section-by-section basis and matching boundary conditions at each transition.

As a second example, we will take $v_0 = 5$,

$$C = \begin{pmatrix} 0, & 1/2 \\ -1, & 1 \\ 1, & 1/2 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 0, & 1 \\ -1, & 2 \\ 1, & 1 \end{pmatrix}$$

from which

$$A = CR = \begin{pmatrix} 0, & 1/2 \\ -1/2, & 1 \\ -2, & 2 \\ 0, & 1 \\ 3/2, & 1/2 \end{pmatrix} \begin{matrix} 1 \\ 3/2 \\ 2 \\ 3 \end{matrix}$$

After calculating B from Eqs. (221), (224), and (223), we can write the solution in the form

$$v = \begin{pmatrix} 0, & 1 \\ -1, & 2 \\ 1, & 1 \end{pmatrix} \left[\exp \begin{pmatrix} 3.0602, & 2 \\ 1.5301, & 1 \\ 0.9678, & \tfrac{1}{2} \\ 1.2425, & 1 \\ 1.3863, & 2 \end{pmatrix} (-t) \right] \qquad (226)$$

Equation (226) is equivalent to the set of equations

$$v = 1 + \exp(1.3863 - 2t) = 1 + 4e^{-2t} \qquad (0 < t < t_1)$$

$$v = 1 + \exp(1.2425 - t) = 1 + 3e^{-(t-t_1)} \qquad (t_1 < t < t_2)$$

$$v = -1 + 2\exp(0.9678 - \tfrac{1}{2}t) = -1 + 4e^{-\frac{1}{2}(t-t_2)} \qquad (t_2 < t < t_3)$$

$$v = -1 + 2\exp(1.5301 - t) = -1 + 3e^{-(t-t_3)} \qquad (t_3 < t < t_4)$$

$$v = 0 + \exp(3.0603 - 2t) = e^{-2(t-t_4)} \qquad (t_4 = t)$$

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where

$$t_1 = 0.1438 \qquad t_2 = 0.5493 \qquad t_3 = 1.1246 \qquad t_4 = 1.5301$$

The principal advantage of using PWL operators instead of conventional methods in these examples is not that the total amount of computation is appreciably reduced but rather that the work is much more systematic.

B. GENERAL PWL R-C NETWORKS

In this section we will attempt to generalize the method used to analyze the parallel PWL R-C network to more general PWL networks that may contain any number of PWL resistors and capacitors.[*] We will show that the voltages in such networks can be expressed in the form

$$v = \sum_j A_j \left\{ \exp [B_j (-t)] \right\} \tag{227}$$

If all of the PWL R's and C's have finite, positive slopes, at any instant of time the network reduces to a network composed of linear resistors and capacitors and d-c sources. Therefore, during any interval of time, the voltage at any point in the network must consist of a sum of decaying exponentials plus a d-c term. The voltage at any point can therefore be described by the equations

$$v = \sum_{j=1}^{m} a_{jk} \exp (-b_{jk} t) + a_{ok} \qquad (t_{k-1} < t < t_k)$$
$$(k = 1, 2, \dots, n) \tag{228}$$

Since all of the currents in the network are finite, none of the voltages across the capacitors can change instantaneously, and all of the voltages must be continuous at the transition points, i.e.

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[*] The results of this analysis can, of course, be applied to the general PWL R-L network through the use of duality.

$$\sum_{j=1}^{m} a_{jk} \exp(-b_{jk} t_k) + a_{ok} = \sum_{j=1}^{m} a_{j,k+1} \exp(-b_{j,k+1} t_k) + a_{o,k+1}$$

$$(k = 1,2,\ldots,n-1) \qquad (229)$$

We will now show that Eq. (228) can be expressed in the following form:

$$v = \sum_{j=1}^{m} \begin{pmatrix} q_{j1}, & r_{j1} \\ q_{j2}, & r_{j2} \\ \vdots \\ q_{jn}, & r_{jn} \end{pmatrix} \left[\exp \begin{pmatrix} c_{j1}, & d_{j1} \\ c_{j2}, & d_{j2} \\ \vdots \\ c_{jn}, & d_{jn} \end{pmatrix} (-t) \right] \qquad (230)$$

During any interval of time, Eq. (230) reduces to

$$v = \sum_{j=1}^{m} [q_{jk} + r_{jk} \exp(c_{jk} - d_{jk} t)] \qquad (t_{k-1} < t < t_k) \qquad (231)$$

where

$$t_k = \frac{c_{j,k+1} - c_{jk}}{d_{j,k+1} - d_{jk}} \qquad (232)$$

In order that Eq. (231) be equivalent to Eq. (228), the following relations must hold for all j and k :

$$b_{jk} = d_{jk} \qquad (233)$$

$$a_{jk} = r_{jk} \exp(c_{jk}) \qquad (234)$$

$$a_{ok} = \sum_{j=1}^{m} q_{jk} \qquad (235)$$

For $k = 1$, we can satisfy Eqs. (234) and (235) by taking

$$q_{11} = a_{01} \qquad\qquad q_{j1} = 0 \qquad (j > 1) \qquad\qquad (236)$$

$$c_{j1} = 0 \qquad (\text{all } j) \qquad r_{j1} = a_{j1} \qquad (\text{all } j) \qquad\qquad (237)$$

Once we have chosen c_{j1}, the remaining c_{jk}'s must be chosen so that Eq. (232) is satisfied, i.e.,

$$c_{j,k+1} = (d_{j,k+1} - d_{jk}) t_k + c_{jk} \qquad (k = 1,2,\ldots,n\text{-}1) \qquad (238)$$

Since the transitions between successive sections of both PWL operators in Eq. (230) must occur at the same time, the breakpoints must match; this matching requires that

$$\frac{q_{j,k+1} - q_{jk}}{r_{jk} - r_{j,k+1}} = \exp\,(c_{jk} - d_{jk}\, t_k) = \exp\,(c_{j,k+1} - d_{j,k+1}\, t_k) \qquad (239)$$

from which

$$q_{j,k+1} = r_{jk}\, \exp\,(c_{jk} - d_{jk}\, t_k) - r_{j,k+1}\, \exp\,(c_{j,k+1} - d_{j,k+1}\, t_k)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (240)$$
$$+ q_{jk}$$

We have now evaluated all of the q's, r's, c's, and d's in Eq. (230). To verify the equivalence between Eqs. (231) and (228), the only thing that remains to be done is to show that the q's defined by Eq. (240) satisfy Eq. (235). Substituting Eqs. (233) and (234) into Eq. (240) and then summing over j, we obtain

$$\sum_j q_{j,k+1} = \sum_j a_{jk}\, \exp\,(-b_{jk}\, t_k) - \sum_j a_{j,k+1}\, \exp\,(-b_{j,k+1}\, t_k) + \sum_j q_{jk} \qquad (241)$$

By using Eq. (229), Eq. (241) can be reduced to

$$\sum_j q_{j,k+1} = \Big(\sum_j q_{jk} - a_{ok}\Big) + a_{o,k+1} \qquad\qquad (242)$$

If Eq. (235) is satisfied for a given value of k, it follows from Eq. (242) that it is also satisfied for k+1 . But from Eq. (236)

$$\sum_j q_{j1} = a_{01},$$ and Eq. (235) is satisfied for k = 1. It therefore follows by induction that Eq. (235) is satisfied for all values of k .

This completes the proof that the voltage at any point in a PWL R-C network can be expressed in the form of Eq. (227). This proof does not provide a convenient way of determining the A_j's and B_j's, but it at least shows the existence of a solution in terms of PWL operators.

The voltages in a PWL R-C network must satisfy a set of PWL differential equations. From the above analysis, we know that the solutions to these equations can be expressed in the form of Eq. (225). Hopefully, we could substitute solutions of this form into the differential equations and then solve for the unknown PWL operators. Unfortunately, this procedure does not work very well because the left distributive law cannot generally be used and we encounter difficulties in differentiating the PWL functions.

As an example, we will consider the network of Fig. 58. This network can be described by the differential equations

$$\frac{dv_1}{dt} + (v_1 - v_2) = 0 \qquad (243)$$

$$\frac{dv_2}{dt} + (v_2 - v_1) + \begin{pmatrix} 0, & 1 \\ 1, & 1/2 \end{pmatrix} (v_2) = 0 \qquad (244)$$

From Eq. (243),

$$v_2 = \frac{dv_1}{dt} + v_1 \quad \text{and} \quad \frac{dv_2}{dt} = \frac{d^2 v_1}{dt^2} + \frac{dv_1}{dt} \qquad (245)$$

Substituting into Eq. (244), we obtain

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$$\frac{d^2 v_1}{dt^2} + 2\frac{dv_1}{dt} + \begin{pmatrix} 0, & 1 \\ 1, 1/2 \end{pmatrix} (\frac{dv_1}{dt} + v_1) = \frac{d^2 v_1}{dt^2} + \begin{pmatrix} 0, & 3 \\ 1, 5/2 \end{pmatrix} (\frac{dv_1}{dt} + v_1) - 2v_1 = 0$$

$$(246)$$

We will solve this PWL differential equation using the initial conditions

$$v_1(0) = 0 \qquad v_2(0) = 6 \qquad \dot{v}_1(0) = v_2(0) - v_1(0) = 6$$

From the previous discussion, we know that the solution can be expressed in the form

$$v_1 = A_1 \left\{ \exp[B_1(-t)] \right\} + A_2 \left\{ \exp[B_2(-t)] \right\} \qquad (247)$$
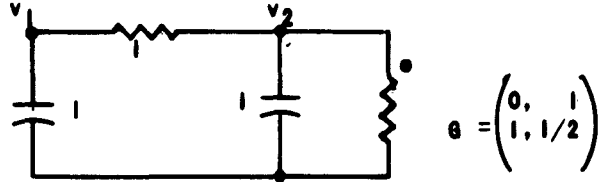
FIG. 58. PWL NETWORK WITH TWO CAPACITORS.

If we substitute Eq. (247) into Eq. (246), we obtain a PWL-operator equation that we do not know how to solve. Our only recourse is to solve Eq. (246) in sections. We can separate Eq. (246) into two linear differential equations:

$$\frac{d^2 v_1}{dt^2} + 5/2 \frac{dv_1}{dt} + 1/2 \, v_1 = -1 \qquad \left(\frac{dv_1}{dt} + v_1 > 2, \ 0 < t < t_1 \right)$$

$$\frac{d^2 v_1}{dt^2} + 3 \frac{dv_1}{dt} + v_1 = 0 \qquad \left(\frac{dv_1}{dt} + v_1 < 2, \qquad t > t_1 \right)$$

These equations have solutions of the form

$$v_1 = c_1 e^{-0.2192t} + c_2 e^{-2.2808t} - 2 \qquad (0 < t < t_1)$$

$$v_2 = c_3 e^{-0.3820t} + c_4 e^{-2.6180t} \qquad (t_1 < t)$$

We can evaluate c_1 and c_2 from the initial conditions and then determine t_1 by solving the equation

$$\dot{v}_1 (t_1) + v_1 (t_1) = 2$$

After evaluating c_3 and c_4 by matching values of $\dot{v}_1$ and v_1 at $t = t_1$, we obtain

$$v_1 = 5.123e^{-0.2192t} - 3.123e^{-2.2808t} - 2 \quad (0 < t < 0.8007)$$

$$v_1 = 2.978e^{-0.382t} - 3.239e^{-2.618t} \qquad (t > 0.8007)$$

This solution can be expressed in terms of PWL operators as

$$v_1 = \begin{pmatrix} 0.105, & 2.614 \\ -2, & 5.123 \end{pmatrix} \left[e^{B_1(-t)} \right] + \begin{pmatrix} -0.105, -2.473 \\ 0, -3.123 \end{pmatrix} \left[e^{B_2(-t)} \right]$$

where

$$B_1 = \begin{pmatrix} 0.13035, & 0.382 \\ 0, & 0.2192 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0.27, & 2.618 \\ 0, & 2.2808 \end{pmatrix}$$

The derivative of v_1 can be expressed in the form

$$\frac{dv_1}{dt} = \begin{pmatrix} -0.105, -0.999 \\ 0, -1.123 \end{pmatrix} \left[e^{B_1(-t)} \right] + \begin{pmatrix} -0.105, & 6.473 \\ 0, & 7.123 \end{pmatrix} \left[e^{B_2(-t)} \right]$$

from which

$$v_2 = \frac{dv_1}{dt} + v_1$$

$$= \begin{pmatrix} 0, & 1.615 \\ -2, & 4 \end{pmatrix} \begin{bmatrix} e^{B_1(-t)} \end{bmatrix} + 4e^{B_2(-t)}$$

In this example, PWL operators were helpful in formulating the differential equations of the network, but they were of no help in solving these equations.

C. PWL L-C NETWORKS

A network that consists of a PWL inductor in parallel with a linear capacitor (Fig. 59a) will now be analyzed. We will assume that the inductor has a symmetrical flux-linkage - current characteristic as shown in Fig. 59b. Assuming no damping, the circuit will oscillate continuously if we start with an initial current in the inductor or an initial charge on the capacitor. It can be shown that, during the first quarter period of oscillation, the current, voltage, charge, or flux linkage can be expressed in terms of PWL operators as

$$x = A_1 \left[\cos B(t) \right] + A_2 \left[\sin B(t) \right] \tag{248}$$

Once the solution for the first quarter period is known, the rest of the solution can be determined from symmetry.

The flux linkage in the network of Fig. 59a satisfies the differential equation

$$c \frac{d^2 \phi}{dt^2} + F(\phi) = 0 \tag{249}$$

where $F(\phi)$ is the current in the inductor. The PWL-inductor characteristic of Fig. 59b is

$$\phi = L(i) = \begin{pmatrix} -27/4, & 1/4 \\ -3, & 1 \\ 0, & 4 \\ 3, & 1 \\ 27/4, & 1/4 \end{pmatrix} (i) \quad \text{or} \quad i = \begin{pmatrix} 27, & 4 \\ 3, & 1 \\ 0, & 1/4 \\ -3, & 1 \\ -27, & 4 \end{pmatrix} (\phi)$$

- 150 -
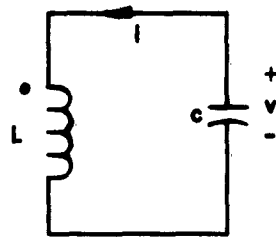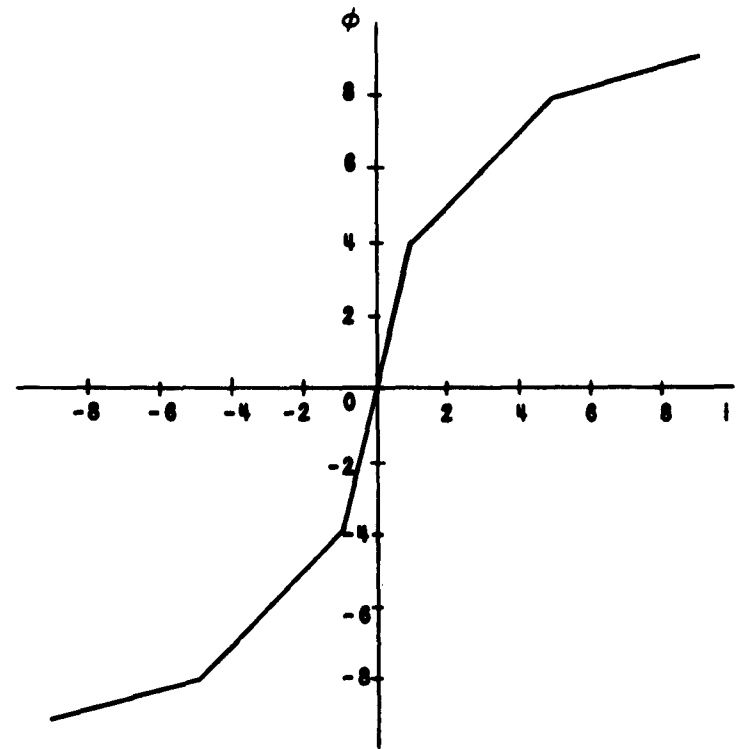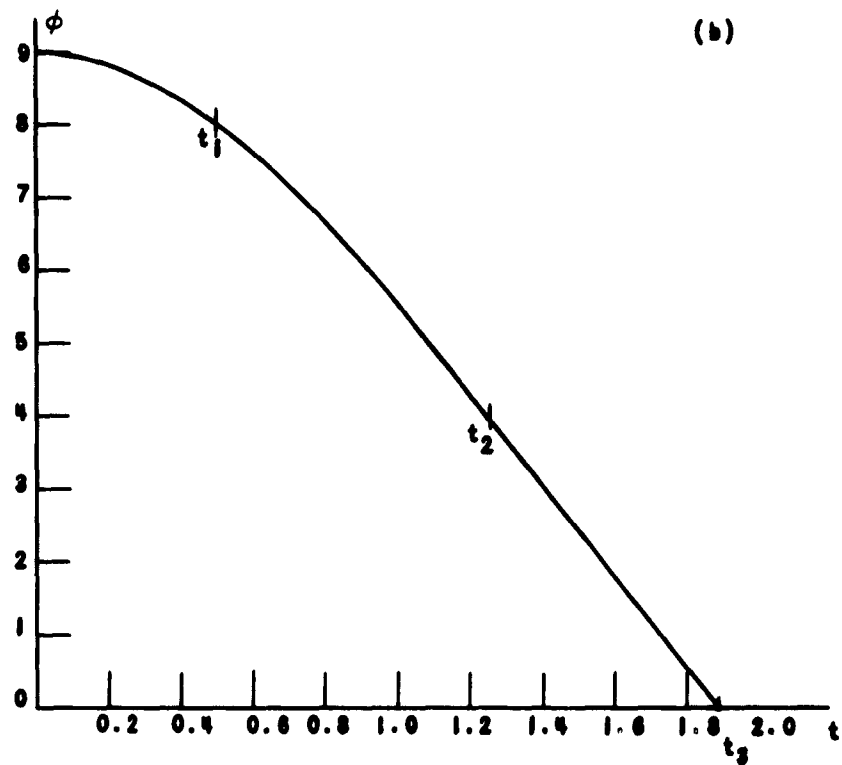
FIG. 59.   PWL L-C NETWORK.

As an example, we will find the solution to Eq. (249) using this charac-
teristic with c = 1, i(0) = 9, and v(0) = 0. For the first quarter
period of oscillation $\phi > 0$, and Eq. (249) becomes

$$\frac{d^2\phi}{dt^2} + \begin{pmatrix} 0, & 1/4 \\ -3, & 1 \\ -27, & 4 \end{pmatrix} (\phi) = 0 \qquad (250)$$

with $\phi(0) = 9$ and $\dot{\phi}(0) = 0$

If we assume a solution of the form Eq. (248) and substitute into
Eq. (250), we run into the usual difficulties with the left distributive
law, so we will solve in sections. Equation (250) can be separated into
three linear differential equations

$$\ddot{\phi} + 4\phi = 27 \qquad (8 < \phi < 9) \qquad (0 < t < t_1)$$

$$\ddot{\phi} + \phi = 3 \qquad (4 < \phi < 8) \qquad (t_1 < t < t_2)$$

$$\ddot{\phi} + \tfrac{1}{4}\phi = 0 \qquad (0 < \phi < 4) \qquad (t_2 < t < t_3)$$

The solutions to these equations are

$$\phi = 6.75 + 2.25 \cos 2t \qquad\qquad (0 < t < t_1)$$

$$\phi = 3 + 5.8889 \cos (t+t_1) + 2.0787 \sin (t+t_1) \qquad (t_1 < t < t_2)$$

$$\phi = 11.4226 \cos (\tfrac{1}{2}t+t_1+\tfrac{1}{2}t_2) + 6.1256 \sin (\tfrac{1}{2}t+t_1+\tfrac{1}{2}t_2) \quad (t_2 < t < t_3)$$

where $t_1 = 0.4909$, $t_2 = 1.2585$, and $t_3 = 1.8858$. Expressing this
set of equations in terms of PWL operators, we obtain

$$\phi = \begin{pmatrix} 6.75 & , & 2.25 \\ 4.7284, & 5.8889 \\ 5.7110, & 11.4226 \end{pmatrix} [\cos B(t)] + \begin{pmatrix} 0 & , & 0 \\ -1.7284, & 2.0787 \\ -5.7110, & 6.1256 \end{pmatrix} [\sin B(t)]$$

where

$$B = \begin{pmatrix} 0, & 2 \\ t_1, & 1 \\ t_1 + \frac{1}{2}t_2, & 1/2 \end{pmatrix} = \begin{pmatrix} 0, & 2 \\ 0.4909, & 1 \\ 1.1201, & 1/2 \end{pmatrix}$$

This solution is plotted in Fig. 59c. The current during the first quarter period is

$$i = \begin{pmatrix} 0, & 1/4 \\ -3, & 1 \\ -27, & 4 \end{pmatrix} (\emptyset)$$

$$= \begin{pmatrix} 0, & 9 \\ 1.7284, & 5.8889 \\ 1.1898, & 2.8557 \end{pmatrix} [\cos B(t)] + \begin{pmatrix} 0, & 0 \\ -1.7284, & 2.0787 \\ -1.1898, & 1.5314 \end{pmatrix} [\sin B(t)]$$

As in the general R-C case, PWL operators are helpful in formulating the differential equations and expressing the solutions to PWL L-C networks, but PWL operator methods are of little help in solving the equations. If we could define suitable complex PWL operators, it might be possible to express the solution to the PWL parallel L-C network in the form

$$x = \mathfrak{R}_e \left\{ A*[e^{B*(t)}] \right\}$$

and it might be easier to find A* and B* than to find A_1, A_2, and B in Eq. (248). The fact that solutions to general PWL R-C, R-L, and L-C networks can be expressed in terms of PWL operators holds promise that better methods of solution can be developed.

IX. COMPUTER PROGRAMS FOR SOLUTION OF PWL OPERATOR EQUATIONS

The methods for representing PWL curves by PWL operators and the algebraic operations with PWL operators have been developed with the use of a digital computer in mind. Although the PWL-operator method is useful for hand computation in simple problems, the method is cumbersome for complex problems unless a computer is used. Where iterative solutions are required, a computer is almost imperative because of the large amount of work involved.

Computer programs have been written for the Burroughs 220 Computer for analysis of resistive PWL networks. Subroutines have been written for addition, subtraction, multiplication, inversion, trivolution, and other operations with PWL operators. These subroutines have been used in programs for solution of PWL-network equations.

A. A BRIEF DESCRIPTION OF BALGOL

The PWL-operator programs are written in an algebraic language called BALGOL, which is the Burroughs version of ALGOL. The BALGOL program consists of a series of statements, which are punched on cards and read into the computer. The compiler program for the computer translates these statements into machine-language instructions. After compilation, the data are read into the computer and the program is executed.

Several types of statements are used in the BALGOL program. The assignment statement causes a variable to be set equal to the value of a given expression. For example,

$$X = (A + B.C)/(D + E*2);$$

means compute the value of $(A + B \cdot C)/(D + E^2)$ and then set X equal to this computed value. Note that the equal sign does not have its usual meaning here. The same variable may appear on both sides of the assignment statement. Thus, $X = 2X + 1$; means compute $2X + 1$ using the present value of X, and then change the value of X to this new value. The IF statement indicates that the next statement in sequence is to be executed only if a given condition is true. For example,

IF A LEQ B; δ ;

means that statement δ is to be executed only if $A \leq B$. The FOR state-
ment causes the next statement or group of statements to be executed a
given number of times. For example,

FOR I = (1,1,N); δ ;

means that statement $\S$ is to be executed for $I = 1$, then for $I = 2$,
then for $I = 3$, and so forth, and finally for $I = N$. A statement may
be labelled by preceding it with an identifier or an integer followed
by two dots, e.g.,

A13.. $\S$;

attaches the label A13 to the statement $\S$. The statement

GO A13;

causes the statement labelled A13 to be executed next. For further in-
formation about BALGOL, the reader should consult the manual for the
Burroughs Algebraic Compiler [Ref. 8].

B. SIMPLIFICATION OF PWL OPERATORS

After two PWL operators have been added, subtracted, or multiplied,
if the answer has two successive sections that have the same slope and
intercept, one of the sections is redundant and must be eliminated.
Simplification of PWL operators by elimination of redundant sections is
also an important step in the iterative solution of PWL-operator equa-
tions. As the iterative solution of a PWL-operator equation is carried
out, the order of the PWL operator in the approximate solution tends to
increase with each iteration. However, as the iteration continues,
some pairs of successive sections of the PWL curve will generally ap-
proach the same line segment. When they are sufficiently close, one of
the sections can be eliminated, and so reduce the order of the PWL

operator. Other sections of the PWL curve will eventually become shorter and shorter, and they can also be eliminated. Proper convergence of the iteration may depend on the criterion used for elimination of redundant sections. If the elimination occurs too soon, the accuracy of the solution is impaired and the iteration may not converge to the correct value. If the elimination occurs too late, the order of the PWL operators becomes too large and much unnecessary computation is required.

Two methods for eliminating redundant sections of PWL operators were tried. Using the slope-intercept form, the kth section was eliminated if the differences between the kth slope and the $(k+1)^{th}$ slope and between the kth intercept and the $(k+1)^{th}$ intercept were sufficiently small. This method proved to be unsatisfactory because very short sections were not always eliminated. It is better to use the breakpoint form and to eliminate the kth breakpoint when the distance between this breakpoint and the line joining the two adjacent breakpoints is sufficiently small. The ratio of the perpendicular distance between the point (x_k, y_k) and the line joining the points (x_{k-1}, y_{k-1}) and (x_{k+1}, y_{k+1}) is

$$ s = \left| \frac{x_k (y_{k+1} - y_{k-1}) - y_k (x_{k+1} - x_{k-1}) + x_{k+1} y_{k-1} - x_{k-1} y_{k+1}}{(x_{k+1} - x_{k-1})^2 + (y_{k+1} - y_{k-1})^2} \right| $$

If s is less than a prescribed value of ϵ, the point (x_k, y_k) is sufficiently close to the line and is deleted.

Both the slope-intercept form and the breakpoint form of PWL operators were tried for computer analysis of PWL networks, and it was decided to use the breakpoint form for the following reasons:

1. When iterative procedures are used, simplification in terms of breakpoints works better than simplification in terms of slopes and intercepts.

2. The inversion operation is easier and faster.

3. Scaling is easier in the breakpoint form since a very large range of slopes must be accomodated in the slope-intercept form.

4. Since the output is in terms of points instead of slopes and intercepts, it is easier to plot the results.

5. In breakpoint form, no special code is needed to distinguish the various types of PWL operators.

C. SUBROUTINES FOR WORKING WITH PWL OPERATORS

The BALGOL subroutines that were written for manipulating PWL operators on the Burroughs 220 Computer will now be discussed. A complete listing of these subroutines is given in Appendix C.

TABLE 2.   BASIC SUBROUTINES FOR PWL-OPERATOR ROUTINE.

SUBROUTINE NAME	FUNCTION OF SUBROUTINE	INPUT PARAMETERS TO SUBROUTINE
INP	Read a PWLO from data cards	G5 = loc. where PWLO is to be stored
PRT	Print a PWLO	G5 = loc. of PWLO to be printed
TFR	Transfer a PWLO from one loc. to another	G1 = old loc. of PWLO H = new loc. of PWLO
INV	Invert a PWLO: $D = A^{-1}$	G1 = loc. of A H = loc. of D (H ≠ G1)
ADD	Add 2 PWLO's: $D = A + B$	G1 = loc. of A (G1 ≠ H) G2 = loc. of B (G2 ≠ H) H = loc. of D
SUB	Subtract 2 PWLO's: $D = A - B$	
MUL	Multiply 2 PWLO's: $D = AB$	
ABC	Trivolve 3 PWLO's: $D = A \cdot B \cdot C$	G1 = loc. of A; G2 = loc. of B G3 = loc. of C; H = loc. of D
ADDI	Add identity operator to a PWLO: $D = A + I$	G1 = loc. of A H = loc. of D
SUBI	Subtract identity operator from a PWLO: $D = A - I$	
COMP	Compare A and B	G1 = loc. of A; G2 = loc. of B EPS3 = allowable diff. in y-coordinates

TABLE 3.   AUXILIARY SUBROUTINES FOR PWL-OPERATOR ROUTINE.

SUBROUTINE NAME	FUNCTION OF SUBROUTINE
ASM	Used by ADD, SUB, and MUL subroutines to compare breakpoints of A and B and to compute the breakpoints of D.
SIMPLIFY	Eliminate redundant points from a PWLO.
ORD	Test a PWLO to see if the rows are in standard order.
REORD	Reorder a PWLO if it is not in standard order.
SET	Compute array locations of A, B, and D.
SHIFT	Used by subroutines TFR and INV to relocate a PWLO.

An array of 3200 words in the computer memory is allocated for storage of PWL operators. An nth-order PWL operator in breakpoint form requires $2(n + 1)$ words of storage, so a maximum of 16 PWL operators of order 99 or less can be stored in this array. The location of each PWL operator in the array is designated by an integer from 1 to 16. Since the space occupied by a PWL operator is variable, an additional array of 16 words is provided to store the size of each of these PWL operators.

The basic subroutines which have been written for working with PWL operators are described in Table 2. To use one of the subroutines, the programmer specifies the values of the input parameters followed by the statement ENTER NAME; where NAME is the name of the subroutine. For example, to multiply the PWL operator in location 2 by the PWL operator in location 5 and store the result in location 8, the following calling sequence is used:

$$G1 = 2; \quad G2 = 5; \quad H = 8; \quad \text{ENTER MUL};$$

The auxiliary subroutines listed in Table 3 are used by the basic subroutines and are not used directly by the programmer.

1. Input and Output

The PWL operators that are to be used as input to the program are listed on data cards in breakpoint form in the following format:

$$N \quad x_o \quad y_o \quad x_1 \quad y_1 \quad \cdots \quad x_n \quad y_n$$

where n is the order of the PWL operator and $N = n + 1$ is the number of rows. The calling sequence

$$G5 = L; \quad \text{ENTER INP};$$

is used to read in a PWL operator and store it at location L.

To print the PWL operator which is stored at location L, the following calling sequence is used

$$G5 = L; \quad \text{ENTER PRT};$$

The output appears on the line printer in the following format:

$$\text{COUNT N} \quad x_0 \quad y_0 \quad x_1 \quad y_1 \quad x_2 \quad y_2 \quad x_3 \quad y_3$$

$$x_4 \quad y_4 \quad x_5 \quad y_5 \quad x_6 \quad y_6 \quad x_7 \quad y_7$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$x_{n-1} \quad y_{n-1} \quad x_n \quad y_n$$

COUNT is an integer used for identifying the output.

2. Inversion

To invert the PWL operator stored at L and store the inverse at L_2, the calling sequence

$$G1 = L_1; \quad H = L_2; \quad \text{ENTER INV};$$

is used. The inversion process is easy to carry out on the computer since it is necessary only to recopy the PWL operator with the x's and y's interchanged. After recopying, if the PWL operator is not in standard order, the order of the rows is reversed by interchanging x_0 and y_0 with x_n and y_n, x_1 and y_1 with x_{n-1} and y_{n-1}, etc.

The subroutine ORD is used to check the order of the rows. This subroutine first computes r_1 and r_n, the slopes of the initial and final segments. If $x_1 < x_2$ and $r_1 > r_n$ or if $x_{n-1} < x_n$ and $r_1 < r_n$, reordering is required.

3. Addition, Subtraction, and Multiplication

The subroutines for addition, subtraction, and multiplication of PWL operators have a large section in common, so they will be discussed together. These subroutines will handle multi-valued as well as single-valued PWL operators. The flow chart for these subroutines is shown in Fig. 60. The numbers and letters enclosed in trapezoids refer to labels in the program. The PWL operators being added, subtracted, or multiplied are referred to as A and B, and their sum, difference, or product as D. The coordinates of the Jth breakpoint of A are designated by XA_J and YA_J, the coordinates of the Kth breakpoint of B (of B^{-1} if

FIG. 00   FLOW CHART FOR ADDITION, SUBTRACTION, AND MULTIPLICATION SUBROUTINES.

multiplication is being carried out) by XB_K and YB_K, and the coordinates of the Lth breakpoint of D by XD_L and YD_L. The subscripts J and K can be indexed in either direction. Increasing J (or K) corresponds to moving down the column in A (or B), while decreasing J (or K) corresponds to moving up the column. The previous value of J (or K) is designated by JP (or KP). The program operates in two modes. The normal mode corresponds to moving to the right along the PWL curves, and the reverse mode corresponds to moving to the left. The number of rows in A is NA and the number of rows in B is NB.

After the locations of the PWL operators in the array have been computed, the variable OP is set equal to 0, 1, or 2 to indicate addition, subtraction, or multiplication respectively. The mode and directions for J and K are initially selected according to Table 4. If down is selected for J (or K), J and JP (or K and KP) are initially assigned the values 1 and 2 respectively. If up is selected, the values NA and NA-1 (or NB and NB-1) are assigned instead. When an attempt is made to add, subtract, or multiply two PWL operators that are incompatible, the problem is rejected, and the error message, "REJECT," is printed. The main loop of the subroutine, which begins with the block labeled COMPARE, is traversed repeatedly until all of the breakpoints of A and B have been compared. The values of XD_L and YD_L are computed using the equations shown on Fig. 60. Then J, K, or both are increased or decreased as is appropriate, and a test for corner points is performed. Whenever a corner is detected, the mode of operation is changed. If A has a corner, the direction of K is changed; and if B has a corner, the direction of J is changed. Then L is increased, and before going around the loop again, a check is made to see if the last section of either A or B has been reached. Since the procedure used at the endpoints is different from that at the interior breakpoints, a different path is followed the first and last times through the loop. The first time, a path labeled "FIRST = 1" is followed, and the last time, a path labeled "LAST = 1" is followed. After the last time through the loop, the answer is simplified by deleting redundant points, and the order of the points is reversed if they are not already in conventional order.

TABLE 4. MODE SELECTION.

Type of B*	Mode				J Direction				K Direction			
	Type of A				Type of A				Type of A			
	LR	LL	RL	RR	LR	LL	RL	RR	LR	LL	RL	RR
LR**	N	N	N	R	D	D	U	D	D	D	D	U
LL	N	N	N	X	D	D	U	X	D	D	D	X
RL	N	N	N	R	D	D	U	D	U	U	U	D
RR	R	X	R	R	U	X	D	D	D	X	D	D

N = Normal   R = Reverse   D = Down   U = Up   X = Reject

* When multiplication is being carried out, type of B^{-1} is used instead.

** See Table 1 for explanation of type designations.

4. Trivolution

The subroutine for trivolution uses a modification of the section method described in Section IV. F. If A is in location L_1, B is in L_2, and C is in L_3, the calling sequence

$$G1 = L_1; \quad G2 = L_2; \quad G3 = L_3; \quad H = L_4; \quad \text{ENTER ABC};$$

is used to compute D = A*B*C and store D in location L_4. As in Eq. (93) (p. 79), X_k is computed for each section of C by the equation

$$X_k = [(A - C_k)^{-1} (B - c_k) - I]^{-1}$$

where C_k is the kth section of C and c_k is the slope of C_k. Rather than using Eq. (96) to combine the X_k's to form the final solution, a faster method is used. After each X_k has been computed, the appropriate sections of X_k, which lie between the $(k-1)^{th}$ and the kth breakpoint of C, are selected and combined to form D.

This procedure will always give the correct answer if D is single-valued, but it may give the wrong answer if D is multi-valued. Therefore, after D has been computed, the answer is checked by substitution in the equation $A(D + I) = BD + C$. If the answer fails to check, another attempt to compute D is made by interchanging B and C and

using the equation

$$D = (A*C*B)^{-1}$$

The correct answer is usually obtained on the second try unless D is multi-valued in both directions. If the answer doesn't check the second time, the problem is rejected and the error message "REJECT" is printed.

5. Comparison of PWL Operators

When solving a PWL operator equation by an iterative procedure, a test for convergence of the iteration is needed. The iteration can be terminated when two successive approximations to the solution differ by a sufficiently small amount. The calling sequence

$$G1 = L_1; \quad G2 = L_2; \quad ENTER \ COMP;$$

is used to compare the PWL operator stored in location L_1 with the one stored in L_2. If the difference between the two PWL operators is within the prescribed limit, the variable BCOMP is set equal to 1; otherwise, BCOMP is set equal to 0. This variable can then be tested to determine whether or not the iteration should be terminated.

The PWL operators being compared are first subtracted and the difference is simplified by elimination of redundant points. After simplification, if the two PWL operators being compared are nearly equal, the difference will have the form.

$$\begin{bmatrix} x_1, \ y_1 \\ x_2, \ y_2 \end{bmatrix}$$ if it is defined for all values of the independent variable

or

$$\begin{bmatrix} x_1, \ y_1 \\ x_2, \ y_2 \\ x_3, \ y_3 \end{bmatrix}$$ if it is defined for a semi-infinite range of values of the independent variable.

In the first case, if $|y_1| < \epsilon$, and $|y_2| < \epsilon$, or in the second case, if $|y_1| < \epsilon$, $|y_2| < \epsilon$, and $|y_3| < \epsilon$, the difference between the two PWL opera-

tors being compared is close to zero and the comparison checks. The value of ε that is used depends on the accuracy desired in the final solution.

D. A SAMPLE PROGRAM FOR PWL NETWORK ANALYSIS

A sample program will now be presented to illustrate how the PWL operator subroutines are used to solve PWL-operator equations. The program that was used to derive the input v-i characteristic of the network of Fig. 43 is listed in Table 5, as reproduced from punched cards. The information included in the COMMENT declarations is explanatory material that has no effect on the program.

The program is read into the computer following the PWL subroutines listed in Appendix C. After the PWL operators have been read from data cards, the iteration is carried out using Eqs. (154) and (155), (p. 106) until the comparison between two successive approximation checks. Then the input resistance is computed using Eq. (156). The values of X_k and Y_k after each iteration and the final result are printed out on the line printer.

TABLE 5. PROGRAM FOR ANALYSIS OF FIG. 43.

```
COMMENT   READ IN  R1, -(R2+I), -(R3+I), R4, YO $

P17.. FOR G5=(1,1,5)$  ENTER INP$

    EPS1=EPS2=EPS3=EPS4=0.00005$

COMMENT   START ITERATION $

    FOR COUNT=(1,1,20)$  BEGIN

    COMMENT   COMPUTE X(K+1) $

        G1=1$ G2=2$ G3=5$ H=6$ ENTER ABC$   G5=6$ ENTER PRT$

    COMMENT   COMPUTE Y(K+1) $

        G1=4$ G2=3$ G3=6$ H=7$ ENTER ABC$   G5=7$ ENTER PRT$

    COMMENT   COMPARE Y(K) AND Y(K+1) $

        G1=5$ G2=7$ ENTER COMP$   IF BCOMP$ GO P17A$

    COMMENT   REPLACE Y(K) WITH Y(K+1)$

        G1=7$ H=5$ ENTER TFR   END$

COMMENT   IF COMPARISON CHECKS, COMPUTE INPUT RESISTANCE $

P17A..G1=6$ H=8$  ENTER ADD1$   G1=1$ G2=8$  H=9$  ENTER MUL$

      G1=7$ H=10$ ENTER ADD1$   G1=4$ G2=10$ H=11$ ENTER MUL$

      G1=9$ G2=11$ H=12$ ENTER ADD$   G5=12$ ENTER PRT$
```

Note: On key-punch equipment, the dollar sign is used instead of the semi-colon.

X. CONCLUSIONS

A. APPLICATIONS OF PWL OPERATORS

PWL operators provide a systematic method for analyzing PWL networks that is suitable for use with a digital computer. To analyze a non-linear network, the characteristics of the nonlinear elements are approximated by PWL curves, these curves are represented by PWL operators, the network equations are written in terms of the PWL operators, and these equations are solved to give the desired network characteristic.

PWL operators are most useful for determining the input and transfer characteristics of resistive PWL networks. Series-parallel networks and more general networks that contain two PWL resistors can easily be analyzed in terms of the basic algebraic operations that have been defined for PWL operators. The method has been extended to networks containing three PWL resistors by introducing a new operation called trivolution, and four or more PWL resistors can be handled by using a combination of trivolution and iterative procedures. Since the transient response of R-C, R-L, and L-C PWL networks can be expressed in terms of PWL operators, the use of PWL operators in the analysis of such networks looks promising.

The methods developed for the analysis of resistive PWL networks can be extended to the limiting case of nonlinear networks through the use of appropriate graphical procedures. If one is willing to carry out the operations of addition, subtraction, multiplication, inversion, and trivolution graphically, the PWL-operator method that has been developed can be applied directly to the nonlinear characteristics without first making PWL approximations.

B. COMPARISON WITH OTHER METHODS

The main advantages of the PWL-operator method are that its systematic nature makes it relatively easy to program for a digital computer and that unnecessary computation is kept to a minimum. An important disadvantage is that iterative methods are needed to solve many types of PWL-operator equations.

The breakpoint method and the method of assumed states for analysis of PWL networks require that each PWL circuit element be represented by a circuit model that contains ideal diodes, and the individual states of these ideal diodes must be examined during the course of the analysis. The PWL-operator method avoids the necessity of drawing an ideal-diode model and permits one to work directly in terms of the characteristic curves of the PWL elements. When each PWL characteristic has several sections, considerable work may be saved by using PWL operators. In the breakpoint method, and especially in the method of assumed states, the amount of work required goes up rapidly with the number of diodes. Much unnecessary computation may be required before the final result is obtained unless the problem solver can guess the nature of the solution on the basis of his experience.

In the PWL-operator method, only those computations actually necessary for the final result need be performed. In Stern's method, addition of two PWL characteristics requires that every section of one be added to every section of the other, but in the PWL-operator method, a selection of the sections to be added is made before the addition is carried out. As an example, compare the amount of work involved in the PWL-operator addition of Eq. (30) with the equivalent addition in Stern's notation (Eq. A-9). The superiority of PWL operators in this example is quite apparent. The result obtained in Stern's notation contains nine sections, three of which are redundant.

During the course of the solution of a problem in Stern's notation, the addition process frequently introduces redundant sections. The only method Stern gives for elimination of redundant sections is to sketch the PWL function each time addition is carried out. If the redundant sections are not eliminated as they occur, the algebra becomes needlessly complicated, and the final expression obtained in the analysis of a network that contains n diodes may have 2^n terms. For a function of two or more variables, elimination of redundant sections by sketching the function is clearly impossible, and Stern does not give any method of elimination in this case. Redundant sections are rarely introduced in the addition of PWL operators, and if they are introduced, they are easily eliminated.

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The more concise PWL-operator notation is better suited for a digital computer program than Stern's notation. Stern's method would be difficult to program for several other reasons. In particular, elimination of redundant sections by sketching the function would be difficult. The fact that a given PWL function can be expressed in many different forms in Stern's notation would also cause trouble. The representation of a given PWL function in terms of PWL operators in slope-intercept form is unique. It is much easier to go back and forth between graphical and PWL-operator representation of PWL functions than between graphical and Stern's representation.

Stern does not explicitly state what class of problems can be solved by his method. Just as in the PWL-operator method, once the equations for a problem have been set up, there is no guarantee that they can be solved. There are two places in the solution by Stern's method where we may run into difficulty--the implicit equation theorem and the inversion theorem. The conditions under which these theorems can be applied are stated in Appendix A. Situations arise where the solution of equations written in Stern's notation cannot be completed because the conditions for applying one of these theorems are not satisfied.

Some problems that cannot be solved by PWL operators can be solved by Stern's method and conversely. Some PWL-operator equations that cannot be solved without iteration can be converted to Stern's notation and solved directly. Since multi-valued PWL characteristics cannot be represented explicitly in terms of $\varnothing^-$ transformations, problems that involve this type of characteristic cannot be solved by Stern's method. PWL functions of two variables can be represented more easily in Stern's notation than in PWL-operator notation, and some two-port network problems which cannot be solved by PWL operators can be solved by Stern's method. In problems that can be solved by both methods, the PWL-operator method generally requires less work.

C. SUGGESTIONS FOR FUTURE WORK

This research has created as many new problems as it has solved. Better methods are needed for the solution of PWL-operator equations without the use of iteration. A method for generalizing PWL operators

to represent functions of two variables would facilitate the analysis of PWL two-ports. A more rigorous development of the theory, especially for the case of multi-valued operators, would be helpful. In particular, a rigorous proof that Eq. (63) cannot be solved in terms of the basic algebraic operations is needed. Application of PWL-operator methods to synthesis of PWL networks should be investigated. The most useful, and probably the most difficult, extension of PWL-operator theory would be the development of methods for the solution of PWL differential equations in terms of PWL operators.

ACKNOWLEDGMENT

APPENDIX A: STERN'S METHOD FOR ANALYSIS OF PWL NETWORKS

As discussed in Section II. E., PWL characteristic curves can be expressed in terms of ϕ^+_- transformations. Stern's algebra [Refs. 4,5] for working with these transformations is summarized here for comparison with the PWL-operator method.

ϕ^+_- transformations are defined to transform vectors into scalars. If p is the greatest member of the vector α, and q is the least member, then $\alpha\phi^+ = p$ and $\alpha\phi^- = q$

EXAMPLE: $(1,2,3)\phi^+ = 3$ $(1,2,3)\phi^- = 1$

A vector addition, which is associative and commutative, is defined. The components of the vector sum $\alpha \oplus \beta$ consist of all possible sums of one component of α and one component of β.

EXAMPLE: $(1,2,3) \oplus (4,6,8) = (5,6,7,8,9,10,11)$

The product of a scalar c and a vector $\alpha = (a_1, a_2, \ldots, a_n)$ is $c\alpha = (ca_1, ca_2, \ldots, ca_n)$. Scalar multiplication is distributive with $c(\alpha \oplus \beta) = c\alpha \oplus c\beta$. The following rules are useful in the solution of PWL-network problems:

$$(c\alpha)\phi^+_- = \begin{cases} c(\alpha\phi^+_-) & \text{if} \quad c \geq 0 \\ c(\alpha\phi^-_+) & \text{if} \quad c \leq 0 \end{cases} \qquad \text{(A-1)}$$

$$\alpha\phi^+_- = -[(-\alpha)\phi^-_+] \qquad \text{(A-2)}$$

$$\text{If} \quad \alpha = (a), \qquad \alpha\phi^+_- = a \qquad \text{(A-3)}$$

$$(\alpha \oplus \beta)\,\phi^+_- = \alpha\phi^+_- + \beta\phi^+_- \qquad \text{(A-4)}$$

$$(\alpha\phi^+_-, \beta)\,\phi^+_- = (\alpha, \beta)\,\phi^+_- \qquad \text{(A-5)}$$

The _inversion theorem_ provides a method for inverting PWL functions that are expressed in terms of $\emptyset^-$ transformations. If

$$y = F(x) = [f_1(x),\ f_2(x),\ldots,f_n(x)]\ \emptyset^+_-$$

and each $f_i(x)$ is a continuous, monotonically increasing function, then

$$x = F^{-1}(y) = [f_1^{-1}(y),\ f_2^{-1}(y),\ldots,f_n^{-1}(y)]\ \emptyset^-_+ \qquad (A\text{-}6)$$

If the F_i's are monotonically decreasing, $\emptyset^-_+$ is replaced with $\emptyset^+_-$ in Eq. (A-6).

EXAMPLES:

$$y = (x + 2,\ 2x + 4)\emptyset^+ \text{ has the inverse } x = (y - 2,\ \tfrac{1}{2}y - 2)\emptyset^-$$

$$y = (2x,\ -x + 1)\emptyset^- \quad \text{is not invertible}$$

$$y = [(x + 2,\ 2x + 4)\emptyset^+,\ 3x - 6]\emptyset^- \text{ has the inverse}$$

$$x = [(y - 2,\ \tfrac{1}{2}y - 2)\emptyset^-,\ \tfrac{1}{3}y + 2]\emptyset^+$$

The _implicit equation_ theorem states the conditions under which

$$F(x,y) = [f_1(x,y),f_2(x,y),\ldots,f_n(x,y)]\emptyset^+_- = 0$$

can be solved explicitly for y. If each f_i is continuous and monotonically increasing (or decreasing) in y for any constant x, and each equation $f_i(x,y) = 0$ has an explicit solution $y = g_i(x)$, then

$$y = G(x) = [g_1(x),g_2(x),\ldots,g_n(x)]\emptyset^{-(+_-)} \qquad (A\text{-}7)$$

In particular, the equation

$$F(x,y) = (a_1+b_1x+c_1y,\ a_2+b_2x+c_2y,\ \ldots,\ a_n+b_nx+c_ny)\emptyset^+_- = 0$$

can be solved for y as a function of x if and only if all of the c_i's are non-zero and of the same sign.

EXAMPLES:

$$(x+y+1, \ y-x)\phi^+ = 0 \text{ has the solution } y = (-x-1, \ x)\phi^-$$

$$[(x+y+1, \ y-x)\phi^+, \ \tfrac{1}{2}y+3x-2]\phi^- = 0 \text{ has the solution } y = [(-x-1,x)\phi^-,4-6x]\phi^+$$

Neither of these equations can be solved for x.

The same PWL function may be represented in a number of equivalent forms. For example,

$$(y_1,y_2)\phi^+ + (y_3,y_4)\phi^- = [(y_1+y_3, \ y_1+y_4)\phi^-, \ (y_2+y_3, \ y_2+y_4)\phi^-]\phi^+ \quad \text{(A-8)}$$

In terms of ϕ^+_- transformations, the sum of the PWL curves in Fig. 18 is

$$y = [(\tfrac{x}{2}, \ x+1)\phi^+, \ \tfrac{x}{2}+2]\phi^- + [(-\tfrac{x}{4}+2, \ x-3)\phi^+, \ 2x+11]\phi^-$$

$$= [(\tfrac{x}{2}, \ x+1)\phi^+ + (-\tfrac{x}{4}+2, \ x-3)\phi^+, \ (\tfrac{x}{2}, \ x+1)\phi^+ + (2x+11),$$

$$(\tfrac{x}{2}+2) + (-\tfrac{x}{4}+2, \ x-3)\phi^+, \ (\tfrac{x}{2}+2) + (2x+11)]\phi^-$$

$$= [(\tfrac{x}{4}+2, \ \tfrac{3x}{2}-3, \ \tfrac{3x}{4}+3, \ 2x-2)\phi^+, \ (\tfrac{5x}{2}+11, \ 3x+12)\phi^+,$$

$$(\tfrac{x}{4}+4, \ \tfrac{3x}{2}-1)\phi^+, \ \tfrac{5x}{2}+13]\phi^- \quad \text{(A-9)}$$

By making a sketch of this function, one can see that three of the terms are redundant. Eliminating these terms, we obtain

$$y = [(\tfrac{x}{4}+2, \ \tfrac{3x}{4}+3, \ 2x-2)\phi^+, \ \tfrac{5x}{2}+11, \ (\tfrac{x}{4}+4, \ \tfrac{3x}{2}-1)\phi^+]\phi^- \quad \text{(A-9a)}$$

This result is equivalent to that obtained in Eq. (30). It is interesting to note that although the PWL curve has only five sections, the above expression requires six terms.

APPENDIX B: REPRESENTATION OF PWL CURVES WITH ABSOLUTE VALUES

It is often possible to decompose a complex PWL function into a sum of simpler PWL functions. The absolute-value function is a simple PWL function that is useful for this purpose. Any continuous, single-valued PWL function can be represented in terms of absolute values in the form

$$f(x) = a + bx + \sum_i c_i \, |x - b_i| \qquad\qquad \text{(B-1)}$$

where the b_i's are the breakpoints. Since $|x - b_i|$ changes slope from -1 to +1 at $x = b_i$, the slope of $f(x)$ changes by an amount $2c_i$ at $x = b_i$. Hence, if the slope of $f(x)$ changes from r_{i-1} to r_i at b_i,

$$c_i = \tfrac{1}{2}(r_i - r_{i-1}) \qquad\qquad \text{(B-2)}$$

When x is large and positive,

$$f(x) = a + bx + \sum_i c_i \, (x - b_i)$$

so if the slope of the last section is r_n,

$$b = r_n - \sum_i c_i \qquad\qquad \text{(B-3)}$$

Setting x = 0 gives

$$a = f(0) - \sum_i c_i |b_i| \qquad\qquad \text{(B-4)}$$

As an example, for Fig. 5b,

$$v = -\tfrac{1}{2} + \tfrac{3}{4}i + \tfrac{3}{4} \, |i+4| - \tfrac{1}{2} \, |i+3| - \tfrac{1}{3} \, |i-1| + \tfrac{1}{3} \, |i-4| \qquad \text{(B-5)}$$

PWL functions expressed in terms of absolute values are difficult to manipulate. One can readily appreciate this difficulty if he attempts to solve Eq.(B-5) for i. By going back to the graph of Fig. 5b, it is possible to apply Eqs. (B-2), (B-3), and (B-4) to obtain

$$i = 1 + \frac{3}{2}v - \frac{3}{4}|v+3| + \frac{1}{4}|v+1| + |v-3| - |v-4| \qquad \text{(B-6)}$$

but there does not appear to be any simple, direct method of getting from Eq. (B-5) to Eq. (B-6).

Since the absolute-value function is single-valued, it is impossible to represent multi-valued PWL curves in the form of Eq. (B-1). However, multi-valued curves can often be represented in the more general form

$$a + bx + cy + \sum_i |a_i + b_i x + c_i y| = 0$$

For example, the values of v_2 and i_2 that lie on the curve of Fig. 6d satisfy the equation

$$\frac{3}{2}|v_2 - i_2 - 2| - \frac{3}{2}|v_2 - i_2 + 2| + v_2 - 2i_2 = 0$$

The closed-loop characteristic of Fig. 9d can be expressed as

$$|i_1 - 2v_1| + |i_1| = 2$$

The absolute-value function can be expressed in terms of a PWL operator as

$$|x| = \begin{pmatrix} 0, -1 \\ 0, \ 1 \end{pmatrix}(x) \qquad \text{(B-7)}$$

since both sides of the equation are equal to $-x$ when $x < 0$ and equal to $+x$ when $x > 0$. Any equation that is written in terms of absolute values can be converted to a PWL-operator equation by using Eq. (B-7).

Any PWL function that can be expressed in terms of absolute values can be expressed in terms of ϕ^- transformations and conversely. In an expression that contains absolute values, the relationship

$$|x| = (-x, \ x)\phi^+ \qquad \text{(B-8)}$$

can be used to replace every absolute value with a ϕ^+ transformation. Going from ϕ^- transformations to absolute values is more difficult.

If $x \geq y$, $|x - y| = x - y$ and $\frac{1}{2}(x + y + |x - y|) = x$;

if $x \leq y$, $|x - y| = y - x$ and $\frac{1}{2}(x + y + |x - y|) = y$. Hence

$$\frac{1}{2}(x + y + |x - y|) = (x,y)\emptyset^+ \tag{B-9}$$

and similarly

$$\frac{1}{2}(x + y - |x - y|) = (x,y)\emptyset^- \tag{B-10}$$

Equations (B-9), (B-10), and (24) can be used to convert any expression that contains $\emptyset$ transformations into an equivalent expression that contains absolute values. For example,

$$(x,y,z)\emptyset^+ = [(x,y)\emptyset^+, z]\emptyset^+$$

$$= \frac{1}{2}[\, \frac{1}{2}(x + y + |x - y|) + z + |\frac{1}{2}(x + y + |x - y|) - z|\,]$$

An attempt was made to develop a method for analyzing PWL networks, based on the absolute-value representation for PWL curves. This effort was abandoned because any expression written in terms of absolute values is easily converted to PWL-operator notation or to Stern's notation, both of which are easier to manipulate than absolute values.

APPENDIX C: PWL-OPERATOR SUBROUTINES

The subroutines for working with PWL operators, which are described in Section IX. C., are listed here for reference. These subroutines have been thoroughly tested using various combinations of single- and multi-valued PWL operators. Any of the ten types of PWL operators shown in Fig. 17 may be used as input, with the restriction that infinite slopes are not permitted in the addition, subtraction, multiplication, and trivolution subroutines.

To increase the operating speed of the subroutines, all of the PWL operators are stored in a single one-dimensional array, A, rather than in multi-dimensional arrays. Instead of the notation XA_J, YA_J, XB_K, YB_K, XD_L, and YD_L, which is used in Section IX. C., the notation $A(XA+J)$, $A(YA+J)$, $A(XB+K)$, $A(YB+K)$, $A(XD+L)$, and $A(YD+L)$ is used in the program.

The programs for the problems to be solved should be placed after the subroutines. The first statement in each program should be labelled P1, P2, or P3, etc., and the last statement should be GO NEXT;. The SWITCH statement at the end of the listing, which provides a means for selecting the problem to be solved, should be filled in with the appropriate statement labels.

The reader should refer to the Burroughs Algebraic Compiler Manual [Ref. 8] for a complete explanation of the compiler language.

```
COMMENT  PIECEWISE-LINEAR OPERATOR SUBROUTINESS

FLOATING A( ),T.....,YAP,YBP,EPS...,U,V,Q1,Q2,RC,YC,SLOPES
BOOLEAN B....,NORM,REVJ,REVK,CORNJ,CORNK,ENDJ,ENDK,FIRST,LASTS
INTEGER OTHERWISES
ARRAY A(3800),N(19)$
MA=200$ MB=100$ PROB=0$          EPS1=0.00001$ EPS2=EPS3=EPS4=0.0001$
    GO NEXTS
INPUT DATA(N1,FOR I1=(1,1,N1)$(A(X1+I1),A(Y1+I1)))$
OUTPUT ANS(COUNT,N1,FOR I1=(1,1,N1)$(A(X1+I1),A(Y1+I1)))$
FORMAT FMT(I4,I5,4(X15.6,B1,X11.6),W4,(B9,4(X15.6,B1,X11.6),W0))$
FORMAT HDG1(*REJECT*,W)$  FORMAT HDG2(*L TOO LARGE*,W)$
FORMAT HDG3(*COMPARISON CHECKS*,W)$
FUNCTION SLOPE(X,Y,M)=(A(Y+M+1)-A(Y+M))/(A(X+M+1)-A(X+M))$
FUNCTION BF(U,V)=NORM EQIV (U LSS V)$


SUBROUTINE INPS BEGIN X1=(G5-1)MAS Y1=X1+MBS COUNT=G5+900$
    READ($$DATA)$ N(G5)=N1$ WRITE($$ANS,FMT)$ RETURN ENDS

SUBROUTINE PRTS BEGIN X1=(G5-1)MAS Y1=X1+MBS N1=N(G5)$
    IF ABS(N1) GTR MBS  GO NEXTS  WRITE($$ANS,FMT)$ RETURN ENDS

SUBROUTINE SETS BEGIN XA=(G1-1)MAS YA=XA+MBS XB=(G2-1)MAS YB=XB+MBS
    XD=(H-1)MAS YD=XD+MBS NA=N(H)=N(G1)$ NB=N(G2)$  RETURN ENDS

SUBROUTINE SHIFTS BEGIN FOR I1=(1,1,NA)$ BEGIN
    A(XD+I1)=A(XS+I1)$ A(YD+I1)=A(YS+I1) ENDS  RETURN ENDS

SUBROUTINE TFRS BEGIN ENTER SETS XS=XAS YS=YAS ENTER SHIFTS
    RETURN ENDS

SUBROUTINE INVS BEGIN ENTER SETS XS=YAS YS=XAS ENTER SHIFTS
    X=XDS Y=YDS NS=NAS ENTER REORDS RETURN ENDS

SUBROUTINE ORDS BEGIN B5=(SLOPE(X,Y,1) LEQ SLOPE(X,Y,NS-1))$
    B6=(A(X+1) LEQ A(X+2)) AND NOT B5 OR (A(X+NS-1) LEQ A(X+NS))
        AND B5$    RETURN ENDS

SUBROUTINE REORDS BEGIN ENTER ORDS IF B6S RETURNS NT=NS/2$
    FOR I1=(1,1,NT)$ FOR XS=X,YS BEGIN T=A(XS+I1)$
        A(XS+I1)=A(XS+NS-I1+1)$ A(XS+NS-I1+1)=T ENDS  RETURN ENDS

SUBROUTINE SIMPLIFYS BEGIN M1=1$ M2=2$ FOR M3=(3,1,M4)$ BEGIN
    T1=A(X+M3)-A(X+M1)$  T2=A(Y+M3)-A(Y+M1)$  T3=T1.T1+T2.T2+0.0001$
    IF ABS(A(X+M2).T2-A(Y+M2).T1+A(X+M3).A(Y+M1)-A(X+M1).A(Y+M3))/T3
        GTR EPS2$  (M1=M2$ M2=M2+1)$
    IF M2 NEQ M3$ BEGIN A(X+M2)=A(X+M3)$ A(Y+M2)=A(Y+M3) END ENDS
    N(H)=M2$  RETURN ENDS

SUBROUTINE ADDS BEGIN ENTER SETS OP=0$ B6=B7=1$ ENTER ASMS
    ENTER REORDS RETURN ENDS
SUBROUTINE SUBS BEGIN ENTER SETS OP=1$ B6=B7=1$ ENTER ASMS
    ENTER REORDS RETURN ENDS
SUBROUTINE MULS BEGIN ENTER SETS OP=2$ X=YBS Y=YB=XBS XB=XS  B7=1$
    NS=NBS  ENTER ORDS ENTER ASMS ENTER REORDS RETURN ENDS
```

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```
SUBROUTINE COMP$ BEGIN  H=19$ ENTER SUB$
    IF N(19) GTR 3$ GO COMP1$ IF ABS(A(YD+1)) LSS EPS3$
    IF ABS(A(YD+2)) LSS EPS3$
    IF (N(19) EQL 2) OR (ABS(A(YD+3)) LSS EPS3)$  BEGIN
        WRITE($$HDG3)$ BCOMP=1$ RETURN END$
    COMP1..BCOMP=0$  RETURN END$


SUBROUTINE ASM$ BEGIN
COMMENT  SELECT MODE $
    B1=(A(XA+1) LEQ A(XA+2))$  B2=(A(XA+NA-1) GTR A(XA+NA))$
    B3=(A(XB+1) LEQ A(XB+2))$  B4=(A(XB+NB-1) GTR A(XB+NB))$
    IF NOT B6$ (BTE=B3$ B3=B4$ B4=BTE)$
    IF NOT B7$ (BTE=B1$ B1=B2$ B2=BTE)$
    NORM=B1 AND B3$ IF NOT NORM AND (B2 OR B4)$ GO REJECT$
    REVK=(B3 AND NOT B1) EQIV B6$ REVJ=(B1 AND NOT B3) EQIV B7$
    DJ=DK=J=K=1$ JP=KP=2$  IF REVJ$ (J=NA$ DJ=-1$ JP=NA-1)$
    IF REVK$ (K=NB$ DK=-1$ KP=NB-1)$ L=1$ FIRST=1$ LAST=0$
COMMENT  COMPARE BREAKPOINTS $
COMPARE..IF ABS(A(XA+J)-A(XB+K)) LSS EPS1$ GO ADDBOTH$
    IF BF(A(XA+J),A(XB+K))$ GO ADDJ$ GO ADDK$
ADDJ..YBP=(A(YB+K)-A(YB+KP))(A(XA+J)-A(XB+KP))/(A(XB+K)-A(XB+KP))
      +A(YB+KP)$  SWITCH OP,(SUBJ,MULJ)$  A(YD+L)=YBP+A(YA+J)$
A13..A(XD+L)=A(XA+J)$
    13..IF FIRST$ GO 11$  IF LAST$ GO 15$ JP=J$ J=J+DJ$
    IF BF(A(XA+JP),A(XA+J))$ GO 12$ DK=-DK$ KP=K$ K=K+DK$
MDCH..NORM= NOT NORM$ GO 12$
ADDK..YAP=(A(YA+J)-A(YA+JP))(A(XB+K)-A(XA+JP))/(A(XA+J)-A(XA+JP))
      +A(YA+JP)$  SWITCH OP,(SUBK,MULK)$  A(YD+L)=YAP+A(YB+K)$
A14..A(XD+L)=A(XB+K)$
    14..IF FIRST$ GO 11$ IF LAST$ GO 15$ KP=K$ K=K+DK$
    IF BF(A(XB+KP),A(XB+K))$ GO 12$ DJ=-DJ$ JP=J$ J=J+DJ$ GO MDCH$
ADDBOTH..SWITCH OP,(SUBBOTH,MULBOTH)$  A(YD+L)=A(YA+J)+A(YB+K)$
A11..A(XD+L)=A(XA+J)$
    11..FIRST=0$ IF LAST$ GO 15$ JP=J$ KP=K$ J=J+DJ$ K=K+DK$
COMMENT  TEST FOR CORNER POINTS $
    CORNJ=BF(A(XA+J),A(XA+JP))$  CORNK=BF(A(XB+K),A(XB+KP))$
    IF CORNJ AND CORNK$ GO MDCH$
    IF CORNJ$ (K=K-2DK$ DK=-DK$ GO MDCH)$
    IF CORNK$ (J=J-2DJ$ DJ=-DJ$ GO MDCH)$
12..L=L+1$ IF L GTR MB$ GO LRG$
COMMENT  CHECK END CONDITIONS $
    ENDJ=(J EQL 1) OR (J EQL NA)$ ENDK=(K EQL 1) OR (K EQL NB)$
    IF ENDJ AND ENDK$ (LAST=1$ NORM=NOT NORM$ GO COMPARE)$
    IF ENDJ$ GO ADDK$ IF ENDK$ GO ADDJ$ GO COMPARE$
15..X=XD$ Y=YD$ M4=L$ ENTER SIMPLIFY$ NS=M2$ RETURN$
MULJ..A(XD+L)=YBP$ A(YD+L)=A(YA+J)$ GO 13$
MULK..A(YD+L)=YAP$ A(XD+L)=A(YB+K)$ GO 14$
MULBOTH..A(XD+L)=A(YB+K)$ A(YD+L)=A(YA+J)$ GO 11$
SUBJ..A(YD+L)=A(YA+J)-YBP$ GO A13$
SUBK..A(YD+L)=YAP-A(YB+K)$ GO A14$
SUBBOTH..A(YD+L)=A(YA+J)-A(YB+K)$ GO A11$
REJECT..  WRITE($$HDG1)$
16..X1=XA$ Y1=YA$ N1=NA$ WRITE($$ANS,FMT)$
    X1=XB$ Y1=YB$ N1=NB$ WRITE($$ANS,FMT)$  GO NEXT$
LRG..  WRITE($$HDG2)$ GO 16 END$
```

```
SUBROUTINE ABC$   BEGIN
    B8=0$ GT1=G1$ GT2=G2$ GT3=G3$ HT=H$
COMMENT   COMPUTE ARRAY LOCATIONS $
18..AX=(G1-1)MA$ AY=AX+MB$   BX=(G2-1)MA$ BY=BX+MB$
    NB=N(G1)$ NA=N(G2)$   CX=(G3-1)MA$ CY=CX+MB$ NC=N(G3)-1$
    DX=(H-1)MA$ DY=DX+MB$
    XA=17MA$ YA=BX$ XB=XA+MB$ YB=AX$   XD=18MA$ YD=XD+MB$
    BL=0$ I=0$          M=1$ OP=2$
22..IF BL$ GO 23$ I=I+1$ BL=(I EQL NC)$
COMMENT   COMPUTE ITH PARTIAL ANSWER $
19..Q1=A(CX+I)$ Q2=A(CX+I+1)$ RC=SLOPE(CX,CY,I)$ YC=A(CY+I)-RC.Q1$
    FOR J=(1,1,NB)$ A(XB+J)=A(AY+J)-YC-RC.A(AX+J)$
    FOR K=(1,1,NA)$ A(XA+K)=A(BY+K)-RC.A(BX+K)$
    X=XA$ Y=YA$ NS=NA$ ENTER ORD$ B7=B6$
    X=XB$ Y=YB$ NS=NB$ ENTER ORD$ ENTER ASM$
    FOR J=(1,1,NS)$ A(XD+J)=A(XD+J)-A(YD+J)$
    X=XD$ Y=YD$          ENTER REORD$
COMMENT   MOVE ENDPOINTS OUT $
    IF A(XD+1) GTR Q2-EPS4$   BEGIN
        A(XD+1)=2.0A(XD+1)-A(XD+2)$ A(YD+1)=2.0A(YD+1)-A(YD+2) END$
    IF A(XD+NS) LSS Q2+EPS4$ BEGIN
        A(XD+NS)=2.0A(XD+NS)-A(XD+NS-1)$
        A(YD+NS)=2.0A(YD+NS)-A(YD+NS-1) END$
COMMENT   SELECT SECTIONS FOR ANSWER $
    J=1$ LAST=0$ IF I EQL 1$ GO 21$
20..IF A(XD+J) LSS Q1-EPS4$ (J=J+1$ GO 20)$ IF NOT BL$ GO 25$
21..A(DX+M)=A(XD+J)$ A(DY+M)=A(YD+J)$
24..M=M+1$ IF M GTR MB$ GO LRG$
    J=J+1$ IF J GTR NS$ GO 22$
    IF LAST$ GO 22$     IF BL$ GO 21$
25..IF ABS(A(XD+J)-Q2) LSS EPS4$ (LAST=1$ GO 21)$
    IF A(XD+J) LSS Q2$ GO 21$
    A(DX+M)=Q2$ A(DY+M)=SLOPE(XD,YD,J-1).(Q2-A(XD+J-1))+A(YD+J-1)$
    LAST=1$ GO 24$
23..M4=M-1$ X=DX$ Y=DY$ ENTER SIMPLIFY$
    IF B8$ (G1=H$ H=HT$ ENTER INV)$
COMMENT   CHECK ANSWER $
    G1=HT$ H=19$ ENTER ADDI$   G1=GT1$ G2=19$ H=18$ ENTER MUL$
    G1=GT2$ G2=HT$ H=19$ ENTER MUL$ G1=18$ G2=19$ H=17$ ENTER SUB$
    TEPS2=EPS2$ TEPS3=EPS3$ EPS2=EPS3=0.001$
    G1=17$ G2=GT3$ ENTER COMP$   EPS2=TEPS2$ EPS3=TEPS3$
    IF BCOMP$ RETURN$   G5=HT$ ENTER PRT$
    IF B8$  BEGIN  WRITE($HDG1)$ GO NEXT  END$
    B8=1$ G1=GT1$ G2=GT3$ G3=GT2$ H=17$   GO 18  END$

SUBROUTINE ADDI$ BEGIN ENTER SET$  FOR J1=(1,1,NA)$ BEGIN
    A(XD+J1)=A(XA+J1)$ A(YD+J1)=A(XA+J1)+A(YA+J1) END$ RETURN END$

SUBROUTINE SUBI$ BEGIN ENTER SET$  FOR J1=(1,1,NA)$ BEGIN
    A(XD+J1)=A(XA+J1)$ A(YD+J1)=A(YA+J1)-A(XA+J1) END$ RETURN END$

COMMENT   MAIN PROGRAM STARTS HERE.  SELECT PROBLEM TO BE SOLVED $
    NEXT..PROB=PROB+1$   SWITCH PROB,(P1,P2,P3                  )$
```

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